Lexicographic Majority*

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Abstract

This paper explores a relationship between lexicographic and majority preferences as a novel explanation of preference cycles in choice. Already May (1954) notes that, among subjects in his experiment who did not display a (majority) preference cycle, a vast majority ordered alternatives according to an attribute that they found overridingly important, suggesting that a lexicographic heuristic was used. Our model, Lexicographic Majority, reconciles these findings by providing a unified framework for lexicographic and simple majority preferences. We justify lexicographic majority preferences by providing an axiomatization in terms of behavioral properties.

Keywords: preference cycles, intransitive preferences; lexicographic order; majority order; bounded rationality

JEL codes: D01, D11, D70

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1 Introduction

May (1954) performs an experiment where he asks subjects to choose between pairs of hypothetical marriage partners based on their intelligence, looks and wealth. He notes that, among subjects in his experiment who did not display a (majority) preference cycle, a vast majority ordered alternatives according to an attribute that they found overridingly important, suggesting that a lexicographic heuristic was used. He concludes:

> "What is the significance of this experiment? Of course it does not prove that individual patterns are always intransitive. It does, however, suggest that where choice depends on conflicting criteria, preference patterns may be intransitive unless one criterion dominates."

Thus, already May (1954) seems to suggest an apparent dichotomy between lexicographic and majority preferences. Preference cycles, based on the majority rule, have since May (1954) been well-documented in the literature (see e.g. Russo and Dosher (1983); Zhang, Hsee, and Xiao (2006)). In accordance with May's intuition, the aim of this paper is to propose a model of choice that accommodates (majority) preference cycles, but is flexible enough to allow some attributes to be overridingly important.

To accommodate preference cycles as in May (1954), our model, Lexicographic Majority (LM), allows a non strict ranking of attributes. In the framework of classical lexicographic preferences the decision maker (DM) *strictly* orders attributes and then evaluates alternatives by proceeding sequentially through the ordering. In contrast we allow a DM to assign the same rank to several attributes. The DM then proceeds sequentially through equivalence classes of attributes by comparing alternatives using a majority heuristic restricted to each equivalence class. If a majority of attributes, within an equivalence class, supports an alternative in favor of the other, the DM stops and chooses the alternative.

Majority preferences (May, 1952) and lexicographic preferences (Fishburn, 1975, 1976; Petri and Voorneveld, 2016) are obtained as polar extremes of Lexicographic Majority. If the attribute ranking of the DM is such that no pair of attributes shares a rank, she uses a lexicographic rule. On the other hand, if the agent thinks that all attributes are equally important a majority rule is used. In this way, our theory provides a unified perspective on lexicographic and simple majority preferences.

To illustrate the model, consider as an example an agent that needs to book a hotel room using an online service (such as hotels.com or booking.com). She evaluates hotels with the aid of a website that uses four attributes $\{\text{Price}, \text{Proximity to City}, \text{Customer rating}, \text{Service}\}$. Suppose that each attribute has a value between 1 and 10 and that the DM's ranking over attributes is Price $\succ$ Proximity to City $\sim$ Customer rating $\sim$ Service. Consider two hotels $x = (5.5, 8, 5, 7)$ and $y = (5.1, 6, 8, 8)$. A DM with LM preferences may then reason that she prefers

$$\begin{array}{ccc} x & = & (\ \overbrace{5.5}^{N_1}\quad \overbrace{8\quad 5\quad 7}^{N_2}\ ) \\ \curlywedge & & \\ y & = & (\ 5.1\quad 6\quad 8\quad 8\ ) \end{array}$$

Blue color indicates a tie. Green color a favored attribute and red color an unfavored attribute. There are two equivalence classes of attributes N_1 and N_2.

Figure 1: An illustration of the LM rule applied to hotel x and y.

y to x. Based on price, she is indifferent since she only notices differences in attributes larger than one (our framework allows coordinate relations to be semi-orders). Looking further at other attributes, on the one hand x is closer to the city, but on the other hand y has a better customer rating and service, and since these three attributes are deemed equally important and a majority of them support hotel y, the DM prefers hotel y to x. The comparison of hotel x and y using the LM rule is illustrated in figure 1.

To see how preference cycles arises within our framework, suppose there is also a third hotel z with characteristics $z = (5,7,4,9)$. An agent using an LM rule as above prefers z to y and y to x. Transitivity would imply that z is preferred to x, but since a majority of attributes among $\{\text{Proximity to City}, \text{Customer rating}, \text{Service}\}$ support x in favor of z, it follows that x is preferred to z. The preference cycle above results from the use of a majority rule. A notable feature of our model is that such cycles arises *endogenously*. This is in contrast from standard majority preferences that *necessarily* exhibit such preference cycles, or from lexicographic preferences that rules them out.

The preference rule above is justified by developing an axiomatic model of it. The main representation result is in terms of three simple properties: a weak transitivity requirement, noncompensation and majority robustness. Noncompensation basically says that high levels on some attributes is not enough to compensate for low levels on other attributes. The third axiom, majority robustness, is novel and is a stronger form of robustness in Petri and Voorneveld (2016). It roughly requires this: Suppose x is preferred to y and that a majority of its coordinates indicate that the former is better. Then the decision maker is allowed a change of mind turning one coordinate in favor of x to an indifference: even if one less argument supports the preference, the fact that we started with many arguments in favor of x suggests that such a small change is not enough to give rise to the opposite preference.

As a further justification of the model, we relate it to the important class of weighted majority preferences. It turns out that lexicographic majority preferences exactly coincide with weighted majority rules satisfying majority robustness. Intuitively, majority robustness forces the weight of a low ranked attribute to stay below the sum of weights of higher ranked attributes. An immediate corollary is a characterization of lexicographic preferences as weighted majority rules satisfying majority robustness.

To the best of our knowledge the class of preferences that we study in this paper is new. We are not aware of any study providing direct evidence on the use of lexicographic majority preferences in decision making. However as already mentioned May (1954), Russo and Dosher (1983) and Zhang, Hsee, and Xiao (2006) provide empirical evidence on use of simple majority rules. The work by Russo and Dosher (1983) suggests that the rule is used to save cognitive effort in decision tasks involving evaluation of multiple attributes. Moreover, further indirect evidence on the lexicographic majority rule is in Yee, Dahan, Hauser, and Orlin (2007), and references therein, where evidence on the use of lexicographic preferences is given.

The rest of the paper is organized as follows. The model is formulated in section 2. We state and discuss our axioms in section 3 . The first characterization theorem is in section 4, where we also discuss some corollaries to the result. Section 5 shows that LM accommodates certain preferences cycles observed in choice. In section 6 we discuss an interpretation of lexicographic majority as a search procedure. Section 7 gives the second main result, a characterization in terms of weighted majority preferences. Finally, we conclude in section 8.

2 The Model

In this section we introduce our model. The notation is standard and is in appendix *A*. From here on, $\succ$ (a preference relation) is an asymmetric binary relation on a nonempty product set $X = \times_{i\in I} X_i$ with finitely many (but at least two) attributes $i \in I$. Before giving a formal definition of lexicographic majority preferences we introduce some notation.

Let X be a set. A *partition* is a collection of nonempty subsets of X, denoted $\mathscr{P}$, with generic elements denoted $S, R \in \mathscr{P}$, such that $\bigcup_{S\in\mathscr{P}} S = X$ and $S \cap R = \emptyset$ for all distinct $S, R \in \mathscr{P}$. If for each $i \in I$ an asymmetric relation $\succ_i$ on X_i is given, set $A(x, y) := \{i \in A : x_i \succ_i y_i\}$ for every subset $A \subset I$. $A(x, y)$ is the *support* of x in A (relative to y). Let further $\mathbf{P}(x, y) := \{i \in I : x_i \succ_i y_i\}$ and $\mathbf{P}(y, x) := \{i \in I : y_i \succ_i x_i\}$ for all $x, y \in X$. The set $\mathbf{P}(x, y)$ is the *total support* of x, relative to y, among all attributes in I. We are now ready for the formal definition of LM preferences.

Definition 2.1. A preference relation $\succ$ on a product set $X = \times_{i\in I} X_i$ is *lexicographic majority* (LM) if there are asymmetric relations $\succ_i$ on X_i for all $i \in I$, a partition $\mathscr{P}$ of I and a linear order $<_1$ on $\mathscr{P}$ such that for all $x, y \in X$, $x \succ y$ if and only if there is a $S \in \mathscr{P}$ such that $|S(x, y)| > |S(y, x)|$ and $|R(y, x)| > |R(x, y)|$ for no $R \in \mathscr{P}$ with $R <_1 S$. ◁

The partition $\mathscr{P}$ corresponding to a lexicographic majority relation in definition 2.1, is called the *indifference partition* of $\succ$. By letting $\mathscr{P} = \{\{k\} : k \in I\}$ be the finest partition consisting of only singletons $\{k\}$ we obtain lexicographic preferences as in Fishburn (1975). Simple majority preferences arises by letting $\mathscr{P} = \{I\}$ be the coarsest partition on I.

Informally, a decision maker (DM) with lexicographic majority preferences is an agent using a decision heuristic. She first ranks attributes by allowing ties, in such a way that the ranking induces a partition of the set of attributes. Equally ranked attributes belong to the same part/class. She then compares pairs of elements x, y by proceeding sequentially through the ranking of attributes. If several attributes share the first rank, alternative x is better than y if a majority of attributes support x. If the DM is still indifferent, she proceeds and compares alternatives by considering attributes with rank two. She continues in this fashion until one element dominates the other, within an equivalence class, using the majority heuristic. If the whole list of attributes is exhausted, and equally many attributes support the alternatives, she is considered indifferent between x and y.

In contrast to a lexicographic rule, a decision maker using a LM rule is allowed to be indifferent even though there are attributes in support of the alternatives. If several attributes share a rank, ties are broken using the simple majority relation.

3 Axioms and discussion

Let $\succ$ be an asymmetric binary relation on X. To characterize and identify the preference rules described in section 2, we need to find $\succ_i$ on the coordinate sets X_i for each $i \in I$ such that definition 2.1 holds. Our following lemma shows that these coordinate relations $\succ_i$ for each $i \in I$ are unambiguously defined in terms of the primitive relation $\succ$:

Lemma 3.1. *If $\succ$ is lexicographic majority with partition $\mathscr{P}$ and linear order $<_1$, relations $\succ_i$ in definition 2.1 satisfy, for all $i \in I$ and $x_i, y_i \in X_i$:*

$$x_i \succ_i y_i \Leftrightarrow \text{for all } z_{-i} \in X_{-i} : (x_i, z_{-i}) \succ (y_i, z_{-i}). \tag{1}$$

The proof of lemma 3.1 is in appendix C. Given the primitive relation $\succ$ we define for each $i \in I$ candidate relations $\succ_i$ on X_i by equation (1). We also define sets $\mathbf{P}(x, y)$ and $A(x, y)$ in terms of candidate relations $\succ_i$. Note that lemma 3.1 implies that these relations and sets are well defined for LM preferences.

We now state and briefly discuss the axioms used in our characterization of lexicographic majority preferences:

A1 *Noncompensation*: for all $x, y, w, z \in X$,

$$\big[\mathbf{P}(x,y) = \mathbf{P}(w,z), \mathbf{P}(y,x) = \mathbf{P}(z,w)\big] \quad \Rightarrow \quad \big[x \succ y \Leftrightarrow w \succ z\big].$$

A2 *Restricted negative transitivity*: $\succ$ is negatively transitive on some product set $X' = \times_{i \in I}\{a_i, b_i\} \subset X$ where $a_i \succ_i b_i$ for each $i \in I$.

A3 *Majority robustness*: for all $x, y \in X$ with $|\mathbf{P}(x,y)| > |\mathbf{P}(y,x)|$,

a) if $x \succ y$ then there is an $i \in \mathbf{P}(x,y)$ with $(z_i, x_{-i}) \succsim (z_i, y_{-i})$ for all $z_i \in X_i$.

b) if $x \sim y$ then there is an $i \in \mathbf{P}(x,y)$ with $(z_i, x_{-i}) \sim (z_i, y_{-i})$ for all $z_i \in X_i$.

Following, among others, Fishburn (1976, p. 395) coordinate i is called *essential* if there are $a_i, b_i \in X_i$ with $a_i \succ_i b_i$ and it is assumed throughout that each coordinate is essential.[1]

Noncompensation *A1* is a quite standard axiom in a literature on noncompensatory preferences. (Fishburn, 1975, 1976; Bouyssou and Vansnick, 1986). It highlights the ordinal character of noncompensatory preferences. It should however be noted that there are preferences with a similar ordinal character as noncompensatory preferences that are ruled out by the noncompensation axiom. Particular examples include the class of concordance relations in Bouyssou and Pirlot (2005, 2007).[2]

Restricted negative transitivity *A2* is similar to, but stronger than, restricted transitivity in Petri and Voorneveld (2016). Transitivity requirement *A2* is imposed for several reasons. Firstly, a DM using a lexicographic majority relation ranks attributes by allowing ties. This in particular implies that such a DM uses an order with transitive indifference on attributes. Intuitively, "equally important" is a transitive relation on the set of attributes. Secondly, transitivity of the induced equivalence $\sim$ facilitates our proof technique. We use transitivity of $\sim$ at several places in the proof of theorem 4.1. Thirdly, there are important non-technical reasons to assume restricted negative transitivity compared to negative transitivity on *all* of X. For example it implies that the setting above is general enough to account for intransitive preferences. Similarly to Fishburn (1976) and Petri and Voorneveld (2016) the current setting accommodates intransitive lexicographic semiorders as in Tversky (1969) and Manzini and Mariotti (2012).

[1] For a preference $\succ$ satisfying the noncompensation axiom A1 this is without loss of generality. If coordinate i is not essential then $x \succ y$ if and only if $(z_i, x_{-i}) \succ (z_i, y_{-i})$ for all $z_i \in X_i$. Hence we may consider the restriction of $\succ$ to $\{z_i\} \times X_{-i}$ for some $z_i \in X_i$. Roughly speaking, we may 'forget' about inessential coordinates. All axioms assumed in this paper impose restrictions only in terms of essential coordinates.

[2] See section 5.2 in Bouyssou and Pirlot (2005) for a further discussion around this issue. Essentially, the issue is that noncompensation "ignores" coordinates that are not essential.

Part a) of majority robustness is similar to robustness [3] in Petri and Voorneveld (2016), where an interpretation of robustness is given. The interpretation given there applies to the current setting as well. Basically majority robustness requires that: If a majority of attributes are in favor of x, then a preference of x over y should be *robust* to a small change of mind, turning an attribute in favor of x to an indifference.

By the formulation of majority robustness it is clear that majority robustness implies robustness, and by further inspecting the proof of the main result, it can be checked that majority robustness A3 can be replaced with the weaker robustness axiom in a characterization of LM preferences. However, due to the intuitive appeal of the cardinal condition $|P(x,y)| > |P(y,x)|$, we choose majority robustness in a characterization of LM preferences.

So what is the intuition behind part b) of majority robustness? It is indeed quite similar to that of part a), and roughly requires that for alternatives x and y, such that many attributes are in favor of x and only a few in favor of y, then an *indifference* between x and y is robust to a small change of mind, turning an attribute in favor of x to an indifference.

Assuming decisiveness *A7* (see section B) part *b)* of majority robustness vacuously holds. Since decisiveness is an axiom that we try to relax, we need part *b)* of majority robustness.

It is evident that every lexicographic majority relation satisfies part b) of majority robustness, since if $x \sim y$, then $|S(x,y)| = |S(y,x)|$ for all $S \in \mathscr{P}$, so $|\mathbf{P}(x,y)| = |\mathbf{P}(y,x)|$. Part b) of majority robustness is hence implied by the following decisiveness property:

A4 *Majority decisiveness*: for all $x, y \in X$, if $|\mathbf{P}(x,y)| \neq |\mathbf{P}(y,x)|$ then $x \succ y$ or $y \succ x$.

We could replace b) of majority robustness by axiom A4 in a characterization of lexicographic majority preferences.

4 The characterization theorem

Our first main result, theorem 4.1, characterizes lexicographic majority preferences $\succ$ using logically independent axioms A1, A2 and A3. We also show that the parameters $((\succ_i)_{i\in I}, <_1, \mathscr{P})$ of a lexicographic majority preference are unique. As corollaries to theorem 4.1 we obtain characterizations of lexicographic and simple majority preferences.

Theorem 4.1. *Let $\succ$ be an asymmetric binary relation on $X = \times_{i\in I} X_i$. The preference relation $\succ$ is lexicographic majority with triple $((\succ_i)_{i\in I}, <_1, \mathscr{P})$ if and only if $\succ$ satisfies noncompensation A1, restricted negative transitivity A2 and majority robustness A3. Furthermore the triple $((\succ_i)_{i\in I}, <_1, \mathscr{P})$ is unique.*

[3] Robustness requires that: for all $x, y \in X$ with $\mathbf{P}(y,x) \neq \emptyset$ and $\frac{|\mathbf{P}(x,y)|}{|\mathbf{P}(y,x)|} \geq 2$, if $x \succ y$, then there is an $i \in \mathbf{P}(x,y)$ with $(z_i, x_{-i}) \succsim (z_i, y_{-i})$ for all $z_i \in X_i$.

Necessity and logical independence of the axioms is proven in appendix C.4 and appendix E. The proof of sufficiency is below. It is straightforward, although quite tedious. We give a brief outline: As a preliminary step, given the axioms, we define a weak order $<_0$ on I, by declaring a coordinate i to be less than j if and only if the corresponding basis vector e_i is preferred to e_j.[4] Now, since weak decisiveness is not at our disposal (as is the case for lexicographic preferences), $<_0$ is not necessarily a complete order. However a standard lemma stated in appendix A implies that there is a linear order $<_1$ on the set of equivalence classes $\mathscr{P}$ of $<_0$. The main steps of the proof (step 1-5) then show that $<_1$ and $\mathscr{P}$ coincides with a linear order and partition of a lexicographic majority rule. As a first step it is shown that $\succ$ restricted to pairs of elements e_A and e_B where A and B are subsets of a member of $\mathscr{P}$ coincides with majority preferences. In the second step, with the crucial help of majority robustness, it is shown that the ranking of any pair of vectors e_A and e_B in X', such that no element of A and B belong to the same member of $\mathscr{P}$, is determined by the lexicographically highest ranked set in $\mathscr{P}$ that A or B intersect. Step 3 and step 4 then show that we can extend the lexicographic majority order to situations where both A and B intersect the same set in $\mathscr{P}$. Lastly, in step 5, noncompensation extends the ordering on X' to all of X.

Proof. (Sufficiency) Assume that $X = \times_{i\in I} X_i$ where $|I| = n$. By assumption each set X_i contains two elements $x_i, y_i \in X_i$ such that $x_i \succ_i y_i$ for all $i \in I$. Hence we may define a subset X' of X such that $X' = \times_{i\in I}\{x_i, y_i\}$ and $x_i \succ_i y_i$ for all $i \in I$. For each subset A of I, define an element $e_A \in X'$ by $(e_A)_i = x_i$ if $i \in A$ and $(e_A)_i = y_i$ if $i \in I \setminus A$. If A is singleton we will abuse notation somewhat and write $e_A = e_{\{j\}} = e_j$.

Define a binary relation $<_0$ on I by $i <_0 j$ if and only if $e_i \succ e_j$. Then $<_0$ is a weak order since $<_0$ inherits negative transitivity and asymmetry from $\succ$ on X'. Denote by $\sim_0$ the equivalence relation induced by $<_0$ and set $\mathscr{P} = {}^{I}/_{\sim_0}$. Define $<_1$ on $\mathscr{P} = {}^{I}/_{\sim_0}$ by $S <_1 R$ if and only if $s <_0 r$ for some $s \in S, r \in R$. Then $<_1$ is well-defined and by lemma A.1 it follows that $<_1$ is a linear order on $\mathscr{P}$.

We now show that $((\succ_i)_{i\in I}, <_1, \mathscr{P})$ is a triple of asymmetric relations, a linear order and a partition making $\succ$ a lexicographic majority preference.

STEP 1: Let $S \in \mathscr{P}$ and $A, B \subset S$. We claim that $e_A \succ e_B$ if and only if $|A| > |B|$. It suffices to show that for all $A, B \subset S$:

$$\begin{cases} |A| = |B| \implies e_A \sim e_B, \\ |A| > |B| \implies e_A \succ e_B. \end{cases}$$

[4]The basis vectors e_A for some set $A \subset I$ are defined in the first paragraph of the proof of theorem 4.1.

We first show that $e_A \sim e_B$ if $|A| = |B|$ and $A, B \subset S$. For a subset A of S, denote the set obtained by swapping an element $i \in A$ to an element $j \in S \backslash A$ (if possible) by $A_{\{i \leftrightarrow j\}}$. Now since $e_i \sim e_j$ for all $i, j \in S$, it follows by noncompensation A1 that $e_{A_{\{i \leftrightarrow j\}}} \sim e_A$. Since the cardinalities of B and A are equal we can obtain B from A by performing a finite number of such swappings. It hence follows by transitivity of $\sim$ (which in turn follows by restricted negative transitivity *A2*) that $e_B \sim e_A$.

We now show that $e_A \succ e_B$ if $|A| > |B|$ and $A, B \subset S$. The proof is by induction on $|A| - |B|$. Since $\succ$ satisfies noncompensation A1 it follows that $e_A \succ e_B$ whenever $B \subset A$ and $|A| = |B| + 1$. Now let A, B be arbitrary subsets of S such that $|A| = |B| + 1$. Let $j \notin B$ then by above $e_{B \cup \{j\}} \succ e_B$. Since $|B \cup \{j\}| = |A|$ it follows by the previous paragraph that $e_A \sim e_{B \cup \{j\}}$. Then $e_A \sim e_{B \cup \{j\}}$ and $e_{B \cup \{j\}} \succ e_B$ together with negative transitivity A2 of $\succ$ gives $e_A \succ e_B$.

As induction hypothesis let $k \in \mathbb{N}$ and assume that $e_A \succ e_B$ whenever $|A| = |B| + k$ and $A, B \subset S$. Let A, B be subsets of S such that $|A| = |B| + k + 1$. If no such sets exist the claim follows, otherwise consider the set $A \backslash \{j\}$ for some $j \in A$. Then $|A \backslash \{j\}| = |B| + k$ and hence it follows by the induction hypotheis that $e_{A \backslash \{j\}} \succ e_B$. But $|A| = |A \backslash \{j\}| + 1$ and hence the base case above gives $e_A \succ e_{A \backslash \{j\}}$. Finally, $e_{A \backslash \{j\}} \succ e_B$ and $e_A \succ e_{A \backslash \{j\}}$ together with restricted negative transitivity *A2* gives $e_A \succ e_B$.

STEP 2: For every $R \in \mathscr{P}$ let

$$U(R, <_1) := \bigcup_{\{S \in \mathscr{P} | R <_1 S\}} S \quad \text{and} \quad L(R, <_1) := \bigcup_{\{S \in \mathscr{P} | S <_1 R\}} S.$$

Let $R \in \mathscr{P}$. We will show that $e_B \succ e_A$ for every $A \subset U(R, <_1)$ and $B \subset R$. As a first step we show that $e_k \succ e_A$ for every $A \subset U(R, <_1)$ and $k \in R$.

If $A \subset U(R, <_1)$ and $|A| = 1$, then $A = \{j\}$ for some $j \in I$ and $A \subset S$ for some $S \in \mathscr{P}$ such that $R <_1 S$ and hence by definition of $<_1$ it follows that $e_k \succ e_j = e_A$. The base case thus follows. Assume that the statement is true for some positive integer m, i.e. for every $A \subset I$ such that $|A| = m$ and $A \subset U(R, <_1)$ we have $e_k \succ e_A$. Let A be a set such that $A \subset U(R, <_1)$ and $|A| = m + 1$. If no such set exists the claim follows, otherwise assume by contradiction that $e_A \succeq e_k$, then by axiom A3 it follows that $e_{A \backslash \{i\}} \succeq e_k$ for some $i \in A$. But $|A \backslash \{j\}| = m$ and $A \backslash \{j\} \subset U(R, <_1)$ and hence this contradicts the induction hypothesis. The claim follows.

Now let $A, B \subset I$ be nonempty sets such that $B \subset R$ and $A \subset U(R, <_1)$. By the previous claim we have that $e_j \succ e_A$ for all $j \in B$. The Pareto lemma *D*.1 implies that $e_B \succ e_j$ for all $j \in B$ and hence by negative transitivity A2 we have $e_B \succ e_A$.

STEP 3: Let $A, B \subset I$ be such that there is a $R \in \mathscr{P}$ such that $|A \cap R| > |B \cap R|$ and $A \cap S =$

$B \cap S = \emptyset$ for all $S <_1 R$. We aim to show that this implies that $e_A \succ e_B$.

Without loss of generality assume that A and B are such that $(A \cap R) \supset (B \cap R)$. If not, let A' be such that $|A' \cap R| = |A \cap R|$, $A' \cap R \supset B \cap R$ and $A' \cap (I \backslash R) = A \cap (I \backslash R)$, then step 1 implies that $e_{A'} \sim e_A$ and hence $e_A \succ e_B$ if $e_{A'} \succ e_B$.

Now since $(A \cap R) \supset (B \cap R)$ noncompensation *A1* implies that $e_A \succ e_B$ if $e_{A \backslash (B \cap R)} \succ e_{B \cap (I \backslash R)}$. Thus we may further reduce to the case where $B \cap R = \emptyset$. But then $B \subset U(R, <_1)$ and hence since $A \cap R \subset R$ by step 2 it follows that $e_{A \cap R} \succ e_B$. The Pareto lemma in appendix D.1 gives $e_A \succsim e_{A \cap R}$. Since $e_A \succsim e_{A \cap R}$ and $e_{A \cap R} \succ e_B$ restricted negative transitivity *A2* implies that $e_A \succ e_B$.

STEP 4: We show that if $A, B \subset I$ are such that there is a $R \in \mathscr{P}$ with $|A \cap R| > |B \cap R|$ and $|A \cap S| = |B \cap S|$ for all $S <_1 R$, then $e_A \succ e_B$.

To show this, let $C \subset L(R, <_1)$ be any set such that $|C \cap S| = |B \cap S| = |A \cap S|$ for all $S \in \mathscr{P}$ such that $S <_1 R$. Set $(e'_A)_i = (e_C)_i$ if $i \in L(R, <_1)$ and set $(e'_A)_i = (e_A)_i$ otherwise, similarly define e'_B, then by applying step 1 and restricted negative transitivity *A2* several times (an induction argument) it follows that $e_A \sim e'_A$ and $e_B \sim e'_B$ and hence $e_A \succ e_B$ if $e'_A \succ e'_B$. By noncompensation *A1* we have that $e'_A \succ e'_B$ if $e_{A \backslash L(R,<_1)} \succ e_{B \backslash L(R,<_1)}$. But the sets $A \backslash L(R, <_1)$ and $B \backslash L(R, <_1)$ are such that the premises of step 3 are satisfied, hence by step 3 it follows that $e_{A \backslash L(R,<_1)} \succ e_{B \backslash L(R,<_1)}$. Thus $e_A \succ e_B$.

STEP 5: Let $x, y \in X$ be such that there is $R \in \mathscr{P}$ such that $|R(x, y)| > |R(y, x)|$ and $|S(y, x)| > |S(y, x)|$ for no $S <_1 R$. Pick the smallest $R \in \mathscr{P}$ (according to $<_1$) such that $|R(x, y)| > |R(y, x)|$, then $|S(x, y)| = |S(y, x)|$ for all $S <_1 R$. If we set $A = \{i \in I : x_i \succ y_i\}$ and $B = \{i \in I : y_i \succ x_i\}$, then $|A \cap S| = |B \cap S|$ for all $S <_1 R$ and $|A \cap R| > |B \cap R|$ and hence $e_A \succ e_B$ by step 4. Noncompensation *A1* implies that $x \succ y$.

Conversely if $x \succ y$ then we cannot have $|S(x, y)| = |S(y, x)|$ for all $S \in \mathscr{P}$, since by applying step 1 and transitivity of $\sim$ several times we would have $x \sim y$. Thus $|R(x, y)| \neq |R(y, x)|$ for some $R \in \mathscr{P}$. Let $R \in \mathscr{P}$ be the smallest such R (according to $<_1$). If $|R(x, y)| < |R(y, x)|$, then $y \succ x$ by the previous paragraph, contradicting asymmetry of $\succ$. Hence $|R(x, y)| > |R(y, x)|$.

Uniqueness: Let $\succ$ be an LM preference with linear order $<_1$ on a partition $\mathscr{P}$ and with asymmetric relations $(\succ_i)_{i \in I}$ satisfying definition 2.1. Assume that there is another triple $((\succ'_i)_{i \in I}, <'_1, \mathscr{P}')$ such that definition 2.1 holds. Since $\succ$ is lexicographic majority it follows by lemma 3.1 that $x_i \succ_i y_i \Leftrightarrow$ for all $z_{-i} \in X_{-i} : (x_i, z_{-i}) \succ (y_i, z_{-i})$ and $x_i \succ'_i y_i \Leftrightarrow$ for all $z_{-i} \in X_{-i} : (x_i, z_{-i}) \succ (y_i, z_{-i})$ so we have that $x_i \succ_i y_i$ if and only if $x_i \succ'_i y_i$. Hence $(\succ_i)_{i \in I} = (\succ'_i$

$)_{i\in I}$.

To show that $\mathscr{P} = \mathscr{P}'$ it suffices to note that for any LM preference with partition $\mathscr{P}$ we have for all $i, j \in I$: $e_i \sim e_j$ if and only if $i, j \in R$ for some $R \in \mathscr{P}$. Since $\mathscr{P} = \mathscr{P}'$ uniqueness of $<_1$ follows if we show that for all $S, R \in \mathscr{P}$: $S <_1 R$ if and only if $S <_1' R$. Let $S, R \in \mathscr{P}$ with $S <_1 R$. Then $e_i \succ e_j$ for all $i \in S$ and for all $j \in R$ and thus $S <_1' R$. By a symmetric argument $S <_1' R$ implies $S <_1 R$. Hence $<_1 = <_1'$. □

5 Explaining preference cycles

In the introduction we showed that LM preferences explain well-documented preference cycles in multi criteria decision-making. The goal of this section is to identify conditions on LM preferences resulting in such cycles. Consider the following example adapted from May (1954).

Example 5.1. Consider LM preferences on $X = \times_{i\in I} X_i$ with $X_i = \{1, 2, 3\}$ and $3 \succ_i 2 \succ_i 1$ for all $i \in I$. Assume $\mathscr{P} = \{I\}$, i.e. all attributes are equally important. Let three alternatives a, b, c be defined as follows:

$$a = (1, 2, 3),$$
$$b = (3, 1, 2),$$
$$c = (2, 3, 1).$$

Using the LM rule described above results in a preference cycle: $a \succ b \succ c \succ a$. ◁

In an experiment conducted by May (1954) 17 out of 62 subjects displayed a cycle as in example 5.1. Among the 45 subjects who did not display a preference cycle 33 subjects ordered alternatives according to a criterion that they found overridingly important. Thus already May (1954) sees a dichotomy between lexicographic and simple majority preferences. Indeed, as suggested by May's experiment, subjects display dissimilar rankings of alternatives due to subjective differences in $\mathscr{P}$ and $<_0$. Subjects who order alternatives cyclically use a partition $\mathscr{P} = \{I\}$ and consider all attributes equally important, whereas subjects who order alternatives according to an overridingly important attribute, use a partition with $\{i\} \in \mathscr{P}$ for some $i \in I$.

In addition to strict preference cycles, lexicographic majority accounts for preference cycles resulting from intransitivity of indifference $\sim$. Fishburn (1970b) discusses an example similar to example 5.2 below and surveys a literature on preferences with intransitive indifference.

Example 5.2. Let $\succ$ be defined as in example 5.1 and consider the following three alternatives:

$$a = (1,2,1),$$
$$b = (2,1,1),$$
$$c = (1,3,1).$$

Note that $a \sim b$ and $b \sim c$, so transitivity of $\sim$ would imply that $a \sim c$, but $c \succ a$ so $\sim$ cannot be transitive. $\lhd$

Given the examples above it would be desirable to know exactly when a DM with lexicographic majority preferences exhibit such preference cycles. It turns out that the logic of the examples carries over to a quite general setting. Proposition 5.1 roughly shows that a strict preference cycle occurs if and only if a DM regards a pair of "important attributes" as equally important. Similarly a cycle resulting from intransitive indifference occurs if and only if the DM finds two attributes equally important.

Proposition 5.1. *Let $\succ$ be a lexicographic majority relation on product set $X = \times_{i\in I} X_i$ with partition $\mathscr{P}$, linear order $<_0$ on $\mathscr{P}$ and negatively transitive relations $\succ_i$ on X_i for all $i \in I$. Let X be such that*

$$\textit{for all } i \in I, \textit{ there are } a_i, b_i, c_i \in X_i \textit{ with } a_i \succ_i b_i \textit{ and } b_i \succ_i c_i. \tag{2}$$

a) $\succ$ is transitive if and only if $|P| = 1$ for all $P \in \mathscr{P} \backslash \{P^*\}$ and $|P^*| \leq 2$, where $P^* =$ $\max(\mathscr{P}, <_1)$.[5]

b) $\succ$ is negatively transitive if and only if $|P| = 1$ for all $P \in \mathscr{P}$, i.e. if and only if $\succ$ is lexicographic.

Proof. For statement a) a similar construction as in example 5.1 implies that $|P| \leq 2$ for all $P \in \mathscr{P}$. Assume there is $P' \in \mathscr{P}$ with $|P'| = 2$ and $P' <_1 P^*$, i.e. $P' = \{i, j\}$ and $i, j \in I$. Let $k \in P^*$. Since there are $a_i, b_i, c_i \in X_i$ with $a_i \succ_i b_i$ and $b_i \succ_i c_i$ define three alternatives by:

$$x = (b_i, c_j, a_k, x_{-ijk}),$$
$$y = (c_i, a_j, b_k, y_{-ijk}),$$
$$z = (a_i, c_j, c_k, z_{-ijk}),$$

[5]Denote by $\max(\mathscr{P}, <_1)$ the element $P^* \in \mathscr{P}$ such that $P <_1 P^*$ for all $P \in \mathscr{P}$.

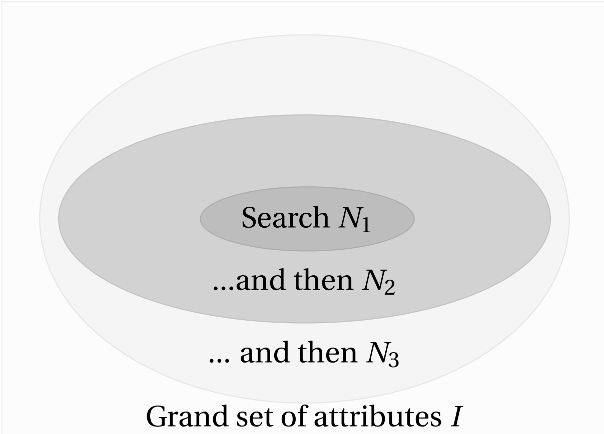

Figure 2: An illustration of the search procedure in proposition 6.1.

where $x_{-ijk} = y_{-ijk} = z_{-ijk}$. Then it is clear that $x \succ y$ and $y \succ z$. But since $z \succ x$ we have a contradiction to $\succ$ being transitive.

The "only if"- part of statement b) follows by the only if part of statement a) and by using a similar construction as in example 5.2. The "if"- parts of statement a) and b) are straightforward to verify and hence omitted. □

6 Lexicographic majority as a search procedure

There is another appealing interpretation of lexicographic majority preferences (suggested to me by Mark Voorneveld) in terms of a decision maker searching over criteria. Let I be a set. A collection of nonempty subsets $\mathcal{N}$ of I is *nested* if $I \in \mathcal{N}$, and if for all $M, N \in \mathcal{N}$ either $M \subset N$ or $N \subset M$. The following proposition gives an equivalent way to express lexicographic majority preferences.

Proposition 6.1. *A preference relation $\succ$ on a product set X is lexicographic majority if and only if there are asymmetric relations $\succ_i$ on X_i for all $i \in I$ and a nested collection of sets $\mathcal{N}$ of I such that for all $x, y \in X$, $x \succ y$ if and only if there is a $N \in \mathcal{N}$ such that $|N(x,y)| > |N(y,x)|$ and $|M(y,x)| > |M(x,y)|$ for no $M \in \mathcal{N}$ with $M \subset N$.*

The proof of proposition 6.1 is straightforward but provided in appendix C for completeness. The proposition above suggests the following alternative procedural interpretation of LM preferences. A decision maker compares alternatives by searching over attributes. First the DM considers a possibly small subset N (as in figure 2) of the grand set of attributes I. If one element dominates the other according to a majority of criteria in N then that element is chosen. Otherwise searching for additional evidence supporting the alternatives enlarges the considered set of attributes. The decision maker hence evaluates alternatives by considering a larger set of attributes $M \supset N$. If one element dominates the other according to a majority of criteria in M then that element is chosen. Otherwise the search continues in a similar manner as before.

7 A characterization in terms of weighted majority preferences

Lexicographic majority preferences is a subset of the class of weighted majority preferences, as proposition 7.1 below shows. In fact, it turns out that lexicographic majority preferences coincide with weighted majority preferences satisfying majority robustness.

Weighted majority preferences are similar to weighted majority rules in social choice theory. A preference relation is weighted majority if x is preferred to y if and only if the sum of weights in support of x is greater than the sum of weights in support of y. The formal definition is as follows:

Definition 7.1. A preference relation $\succ$ on $X = \times_{i \in I} X_i$ is a weighted majority relation if there are weights $w_i \in \mathbb{R}$ for all $i \in I$ such that for all $x, y \in X$:

$$x \succ y \quad \text{if and only if} \quad \sum_{i \in \mathbf{P}(x,y)} w_i > \sum_{i \in \mathbf{P}(y,x)} w_i. \tag{3}$$

◁

Lemma 7.1 below characterizes LM preferences as weighted majority preferences satisfying a weight constraint (4). The necessity of such a constraint is clear. If $\succ$ is a LM rule, then in order for (3) to hold, the weight assigned to a low ranked attribute must be greater than the sum of weights assigned to attributes with a strictly higher (lexicographic) rank. Consider for example the case of purely lexicographic preferences on $\mathbb{R}^n$. Then $(1,0,...,0) \succ (0,1,...,1)$ and representability by a weighted majority rule requires $w_1 > w_2 + ... + w_n$. In lemma 7.1 we show that such a condition is necessary and sufficient for the representability of LM preferences by weighted majority rules. We then apply lemma 7.1 to obtain our main result of this section, theorem 7.2.

Lemma 7.1. *Let $\succ$ be an asymmetric binary relation on product set $X = \times_{i \in I} X_i$. Let $\mathscr{P}$ be a partition of I and let $<_1$ be a linear order on $\mathscr{P}$. The following two claims are equivalent:*

a) $\succ$ is a lexicographic majority preference relation with respect to this ordered partition.

b) $\succ$ is a weighted majority rule with positive weights $(w_i)_{i \in I}$ satisfying (b1) $w_i = w_j$ if i and j belong to the same element of the partition and (b2) for each $i \in I$, if $R \in \mathscr{P}$ is the *partition element containing i, then*

$$w_i > \sum_{j \in \bigcup_{\{S \in \mathscr{P}: R <_1 S\}} S} w_j. \tag{4}$$

Given the lemma above, we obtain the following characterization of lexicographic majority preferences.

Theorem 7.2. *Let $\succ$ be an asymmetric binary relation on $X = \times_{i \in I} X_i$. Then $\succ$ is a weighted majority rule satisfying majority robustness A3 if and only if $\succ$ is a lexicographic majority rule.*

Proof. Let $\succ$ be a weighted majority rule satisfying majority robustness A3 . Define a relation $<_0$ on I by $i <_0 j$ if and only if $w_i > w_j$. It is then straightforward to verify that $<_0$ is a weak order. Let $\sim_0$ on I be the equivalence relation induced by $<_0$ and denote by $\mathscr{P}$ the partition corresponding to $\sim_0$. Further define a relation $<_1$ on $\mathscr{P}$ as in lemma *A*.1. By lemma *A*.1 it then follows that $<_1$ is a linear order on $\mathscr{P}$. By definition of $<_1$ and $\mathscr{P}$ it follows that $w_i = w_j$ whenever $i, j \in S$ for some $S \in \mathscr{P}$. To apply lemma 7.1, and conclude that $\succ$ is a lexicographic majority rule, it hence suffices to show that equation (4) holds. Let $U(R, <_1) := \bigcup_{\{S \in \mathscr{P} | R <_1 S\}} S$ for all $R \in \mathscr{P}$. Assume contrary to equation (4) that there is an $R \in \mathscr{P}$ such that

$$w_i \leq \sum_{j \in U(R, <_1)} w_j$$

for some $i \in R$. Then $e_{U(R,<_1)} \succsim e_i$. By repeatedly applying majority robustness A3 (an induction argument) we get $e_k \succsim e_i$ for some $k \in S$ where $R <_1 S$. But by definition of $<_1$ we have $w_i > w_k$ and hence $e_i \succ e_k$. A contradiction.

The other direction of the theorem follows by lemma 7.1, which shows that every lexicographic majority relation is weighted majority (with weights satisfying equation (4)). Finally, majority robustness of lexicographic majority preferences follow by theorem 4.1. □

Theorem 7.2 implies that the class of weighted majority preferences contains the class of lexicographic majority preferences. However, lexicographic majority preferences and weighted majority preferences are distinct preference structures. Example *E*.3 in appendix E shows that there are weighted majority preferences on sets with three binary factors that fail to satisfy majority robustness. Hence, by characterization theorem 4.1, they cannot be lexicographic majority. Every weighted majority rule satisfies noncompensation A1 and restricted negative transitivity A2. By restricting attention to X with exactly two factors corollary *B*.3 shows that the classes coincide.

LM preferences satisfying weak decisiveness A7 are lexicographic. Hence theorem 7.2 gives us yet another characterization of lexicographic preferences:

Corollary 7.3. *An asymmetric binary relation $\succ$ is lexicographic if and only if $\succ$ is weighted majority and satisfies weak decisiveness A7 and majority robustness A3.*

8 Concluding remarks

We considered lexicographic preferences with a non-strict attribute ranking, using a majority tie breaking heuristic within each indifference class. There are many other plausible tie breaking rules that a decision maker could use. The main technical difficulty, when identifying such preferences is to find properties such that the indifference partition arises endogenously. Also, it is difficult to test a model were the partition is taken as part of the characterizing properties, since it requires the observer to know exactly what partition the decision maker uses when making choices.

It would be interesting to see what other type of heuristics one could obtain without assuming an exogenous indifference partition. In an ongoing project we study a class of preferences called Lexicographic Pareto. Instead of using a simple majority relation within each indifference class, a decision maker with LP preferences uses a Pareto heuristic. It can easily be checked that LM and LP preferences are distinct preference classes: A characterizing property of LM preferences is restricted negative transitivity, and no Pareto relation (on a product set with more than three factors) satisfies this property.

A Appendix: Notation and preliminaries

This section contains standard definitions/notation for binary relations, product sets and partitions. The notation follows, among many others, Fishburn (1975, 1976) and Bouyssou and Vansnick (1986). At the end of this section we state lemma A.1, an important ingredient in the preliminary step of our characterization result.

A.1 Binary relations

A binary relation $\succ$ on a set X is a subset of $X \times X$. If $(x, y) \in \succ$, we write $x \succ y$. For $x, y \in X$, define

$$x \sim y \Leftrightarrow (\text{not } x \succ y \text{ and not } y \succ x) \qquad \text{and} \qquad x \succsim y \Leftrightarrow (x \sim y \text{ or } x \succ y). \tag{5}$$

A binary relation $\succ$ on a set X is: *irreflexive* if, for all $x \in X$: not $x \succ x$, *asymmetric* if, for all $x, y \in X$: $x \succ y$ implies that not $y \succ x$, *transitive* if for all $x, y, z \in X$: $x \succ y$ and $y \succ z$ imply $x \succ z$, *negatively transitive* if for all $x, y, z \in X$: $x \succ y$ implies $x \succ z$ or $z \succ y$, a *weak order* if it is asymmetric and negatively transitive, a *linear order* if it is a weak order, and for all $x, y \in X$ with $x \neq y$, either $x \succ y$ or $y \succ x$.

Asymmetry and negative transitivity imply transitivity: let $x, y, z \in X$ have $x \succ y$ and $y \succ z$. By negative transitivity, $x \succ z$ or $z \succ y$. Asymmetry rules out $z \succ y$, so $x \succ z$. Negative transitivity of $\succ$ implies transitivity of $\sim$: let $x, y, z \in X$ have $x \sim y$ and $y \sim z$. If $x \nsim z$ then w.l.o.g. $x \succ z$ and by negative transitivity $x \succ y$ or $y \succ z$, a contradiction.

A.2 Sets and product sets

$\mathbb{R}$ is the set of real numbers. For subsets A and B of a set I ($A \subset I, B \subset I$), we write $A \setminus B = \{a \in A : a \notin B\}$ and denote the complement of A w.r.t. I as $I \setminus A$ or A^c if I is evident from the context. $|A|$ is the cardinality/number of elements of a finite set A.

For each i in a nonempty index set I, let X_i be a set. Denote their product set by $X = \times_{i \in I} X_i = \{(x_i)_{i \in I} : x_i \in X_i \text{ for each } i \in I\}$. As usual, we refer to elements $i \in I$ as indices, coordinates, or attributes. Conventional notational shortcuts are used. For instance, for $i \in I$, $X_{-i} = \times_{j \in I \setminus \{i\}} X_j$. Let $i, j \in I$ and $A \subset I$ with $i \in A$. Elements $x = (x_k)_{k \in I} \in X$ may be denoted by (x_i, x_{-i}) or (x_i, x_j, x_{-ij}) or $(x_A, x_{-A}) = (x_{A \setminus \{i\}}, x_i, x_{-A})$ if we want to stress coordinates i, j, or those in A.

A.3 Partitions and equivalence relations

A binary relation $\sim$ on X is an *equivalence relation* if $\sim$ is reflexive, symmetric [6] and transitive. The equivalence class of $x \in X$ is defined as $E(x) := \{y \in X : y \sim x\}$. We will use frequently, without proof the following well known fact about equivalence relations and partitions: Every equivalence relation gives rise to a partition in the sense that $\mathscr{P} = \{E(x) : x \in X\}$, and conversely given a partition of X there is an equivalence relation $\sim$ on X defined by $x \sim y$ if and only if $x, y \in S$ for some $S \in \mathscr{P}$, and moreover $\sim$ is such that $\mathscr{P} = \{E(x) : x \in X\}$.

If $\succ$ is a weak order on X then $\sim$ as defined in equation (5) is an equivalence relation. Given a weak order $\succ$ we define a binary relation $\succ'$ on ${}^X/_\sim$, for all $a, b \in {}^X/_\sim$ by $a \succ' b$ if and only if $x \succ y$ for some $x \in a, y \in b$. We call $\succ'$ the *induced order* on ${}^X/_\sim$. Clearly, $\succ'$ is well defined. The following basic result is stated without proof (for a proof see Fishburn (1970a, p. 13, Theorem 2.1))

Lemma A.1. *Let $\succ$ be a weak order on a set X and let $\sim$ be the corresponding equivalence relation, then the induced order $\succ'$ on ${}^X/_\sim$ is a linear order.*

Remark: What we call a linear order, Fishburn (1970a) calls a strict order.

B Appendix: Characterizations of special cases

We will spend this section on characterizing special cases: standard lexicographic preferences and simple majority preferences. We use three additional axioms in our characterizations of lexicographic and simple majority preferences. A permutation $\pi : I \to I$ is a bijective function. Given a permutation $\pi : I \to I$ and $x \in \times_{i \in I} X_i$, where $X_i = X_j$ for all $i, j \in I$, let πx denote the element obtained after permuting the coordinates of x using π, that is $\pi x := (x_{\pi(1)}, ..., x_{\pi(n)})$. In the statement of axiom A6 we assume that $X_i = X_j$ for all $i, j \in I$ since otherwise we may have $\pi x \notin X$.

A5 *Weak anonymity:* for each $i \in I$, let $a_i, b_i \in X_i$ be such that $a_i \succ_i b_i$ and let $b = (b_i)_{i \in I}$. Then for all $i, j \in I$: $(a_i, b_{-i}) \sim (a_j, b_{-j})$.

A6 *Anonymity*: for every permutation $\pi : I \to I$ and for all $x, y \in X$, it holds that $x \succ y$ if and only if $\pi x \succ \pi y$.

A7 *Weak decisiveness:* for all $x, y \in X$, if $x_i \succ_i y_i$ and $y_j \succ_j x_j$ for some distinct $i, j \in I$, then $x \succ y$ or $y \succ x$.

[6] Relation $\sim$ is reflexive if for all $x \in X$: $x \sim x$. Relation $\sim$ is symmetric if for all $x, y \in X$: $x \sim y$ implies $y \sim x$.

Anonymity is commonly assumed in axiomatizations of majority rules in social choice theory (see for instance May (1952)). In order for anonymity to make sense in our setting, we have to impose the condition that coordinate sets are equal ($X_i = X_j$ for all $i, j \in I$). We also need to assume that all coordinate relations are equal ($\succ_i = \succ_j$ for all $i, j \in I$) since otherwise there are LM relations that fail to satisfy anonymity:

Example B.1. Take $X = \{a, b, c, d\} \times \{a, b, c, d\}$. Define asymmetric relations $\succ_1$ and $\succ_2$ on $\{a, b, c, d\}$ such that $a \succ_1 b$ and $c \succ_1 d$, and $a \succ_2 b$ and $d \succ_2 c$. Let $\succ$ be a simple majority relation on X with coordinate relations $\succ_1$ and $\succ_2$. Then $(a, c) \sim (b, d)$ and $(c, a) \succ (d, b)$. But if $(c, a) \succ (d, b)$ then anonymity requires that $(a, c) \succ (b, d)$. A contradiction. ◁

Weak decisiveness is a weak version of decisiveness in Fishburn (1976). Weak decisiveness is introduced in Petri and Voorneveld (2016) where it is used in a characterization of lexicographic preferences.

The corollary below implies that a lexicographic majority relation $\succ$ is lexicographic if and only if $\succ$ is decisive (satisfies *A*7).

Corollary B.1. *Let $\succ$ be an asymmetric binary relation on $X = \times_{i\in I} X_i$. Then $\succ$ is lexicographic if and only if it satisfies noncompensation A1, weak decisiveness A7, restricted negative transitivity A2, and majority robustness A3.*

Proof. If $\succ$ is decisive then each equivalence class of $\sim_0$, in the proof of theorem 4.1, is singleton. By theorem 4.1 it follows that $\succ$ is lexicographic. The converse is immediate. □

Part a) of the next corollary implies that a lexicographic majority relation $\succ$ is simple majority if and only if $\succ$ satisfies weak anonymity *A*5. By adding the hypothesis that all coordinate sets and all coordinate relations are equal part b) of corollary B.2 gives a characterization of simple majority in terms of a standard anonymity axiom.

Corollary B.2. *Let $\succ$ be an asymmetric binary relation on $X = \times_{i\in I} X_i$:*

a) *Then $\succ$ is simple majority if and only if it satisfies noncompensation A1, restricted negative transitivity A2, and weak anonymity A5.*

b) *If in addition $X_i = X_j$ and $\succ_i = \succ_j$ for all $i, j \in I$, then $\succ$ is simple majority if and only if it satisfies noncompensation A1, restricted negative transitivity A2, and anonymity A6.*

Proof. By assumption each set X_i contains two elements $x_i, y_i \in X_i$ such that $x_i \succ_i y_i$ for all $i \in I$. Hence we may define a subset X' of X such that $X' = \times_{i\in I}\{x_i, y_i\}$ and $x_i \succ_i y_i$ for all $i \in I$. For each $i \in I$, define an element $e_i \in X'$ by $(e_i)_i = x_i$ and $(e_i)_j = y_j$ if $j \in I \backslash \{i\}$.

(Part a)) Weak anonymity says that $e_i \sim e_j$ for all $i, j \in I$. By inspecting the proof of theorem 4.1, we see that $\sim_0$ has only one equivalence class, and following step 1 in the proof of theorem 4.1 shows that $\succ$ is simple majority. The converse follows by the necessity part of theorem 4.1 and by noting that $\succ$ satisfies weak anonymity *A*5.

(Part b)) It suffices to show that $e_i \sim e_j$ for all $i, j \in I$, since the result then follows by part a) above. Let $i, j \in I$ and assume that $e_i \succ e_j$. Anonymity *A*6 implies that $\pi e_i \succ \pi e_j$ for all permutations π. Let $\pi : I \to I$ be a transposition such that $\pi(i) = j$,$\pi(j) = i$ and $\pi(k) = k$ for all $k \in I \setminus \{i, j\}$. Then $e_j = \pi e_i \succ \pi e_j = e_i$. A contradiction to $e_i \succ e_j$ and asymmetry of $\succ$. Thus $e_i \sim e_j$ for all $i, j \in I$. □

In corollary *B*.2 majority robustness is not needed to characterize simple majority rules. The axioms in theorem 4.1 are necessary, but not sufficient, for a preference relation to be simple majority. None of the axioms used in the characterization precludes $e_i \succ e_j$ or $e_j \succ e_i$ for some $i, j \in I$, which is why we need (weak) anonymity in corollary B.2. This is contrary to lexicographic preferences that are such that $e_i \succ e_j$ or $e_j \succ e_i$ for all $i, j \in I$. Therefore decisiveness A7 is assumed in corollary B.1. In a characterization of lexicographic preferences robustness is needed: it is logically independent of the other axioms.

It is straightforward to verify that part a) and part b) of majority robustness follows from Pareto lemma D.1 whenever $X = X_1 \times X_2$. Also there are only two choices for the indifference partition in theorem 4.1: either $\mathscr{P} = \{\{1,2\}\}$ or $\mathscr{P} = \{\{1\},\{2\}\}$. We have the following corollary:

Corollary B.3. *Let $\succ$ be an asymmetric binary relation on $X = X_1 \times X_2$. Relation $\succ$ is either lexicographic or simple majority if and only if $\succ$ satisfies noncompensation A1 and restricted negative transitivity A2.*

C Appendix: Proofs

C.1 Proof of lemma 3.1

Proof. Asymmetry of $\succ$ implies $x \sim x$ for all $x \in X$. If $\succ$ is lexicographic majority then by asymmetry of $\succ_i$ for all $i \in I$ it follows that $x_i \sim_i x_i$ for all $i \in I$ and $x_i \in X_i$.

$\Rightarrow$: Let $i \in I, x_i, y_i \in X_i$ have $x_i \succ_i y_i$. Let $z_{-i} \in X_{-i}$. Since $x_i \succ_i y_i$ and $z_j \sim_j z_j$ for all $j \neq i$, we have $|R((x_i, z_{-i}),(y_i, z_{-i}))| = 1 > 0 = |R((y_i, z_{-i}),(x_i, z_{-i}))|$, where $R \in \mathscr{P}$ is the unique R such that $i \in R$. Since $|S((x_i, z_{-i}),(y_i, z_{-i}))| = 0 = |S((y_i, z_{-i}),(x_i, z_{-i}))|$ for all $S \in \mathscr{P}$ with $S \neq R$, lexicographic majority preference gives $(x_i, z_{-i}) \succ (y_i, z_{-i})$.

$\Leftarrow$: Let $i \in I, x_i, y_i \in X_i$, and $z_{-i} \in X_{-i}$ with $(x_i, z_{-i}) \succ (y_i, z_{-i})$. Since $z_j \sim_j z_j$ for all $j \neq i$, lexicographic majority preferences must come from a difference in coordinate i: $x_i \succ_i y_i$. □

C.2 Proof of proposition 6.1

Proof. Let $\succ$ be LM with partition $\mathscr{P}$ and linear order $<_1$ on $\mathscr{P}$. Since $\mathscr{P}$ is a finite set and $<_1$ is a linear order we may without loss of generality assume that $\mathscr{P} = \{P_1, ..., P_k\}$ for some $k \in \mathbb{N}$ and such that $P_i <_1 P_j$ if and only if $i < j$. Now for every $i \in \{1, ..., k\}$ set $N_i := \bigcup_{l=1}^{i} P_l$. Let $\mathscr{N} := \{N_i : i \in \{1, ..., k\}\}$. Then clearly $\mathscr{N}$ is a nested sequence: if $M, N \in \mathscr{N}$, then $M = N_i$ and $N = N_j$ for some $i, j \in \{1, ..., k\}$. If $j < i$ then $N_j = \bigcup_{l=1}^{j} P_l \subset \bigcup_{l=1}^{i} P_l = N_i$, and similarly if $i < j$ then $N_i \subset N_j$. Also $I = \bigcup_{l=1}^{k} P_l = N_k \in \mathscr{N}$.

Assume $x \succ y$, then there is an $i \in \{1, ..., k\}$ such that $|P_i(x, y)| > |P_i(y, x)|$ and $|P_j(x, y)| = |P_j(y, x)|$ for all $j < i$. By definition of $\mathscr{N}$ this implies that $|N_i(x, y)| - |N_i(y, x)| = \sum_{j \le i} \big(|P_j(x, y)| - |P_j(y, x)|\big) = |P_i(x, y)| - |P_i(y, x)| > 0$. If $M \subset N$ then $M = N_j$ for some $j < i$, and a similar calculation then shows that $|M(x, y)| - |M(y, x)| = 0$.

If $x \sim y$ then $|P_i(x, y)| = |P_i(y, x)|$ for all $i \in \{1, ..., k\}$ and hence $|N_i(x, y)| = |N_i(y, x)|$ for all $i \in \{1, ..., k\}$.

Conversely let $\mathscr{N}$ be a nested sequence. Then w.l.o.g. there is a $k \in \mathbb{N}$ such that $\mathscr{N} = \{N_1, ..., N_k\}$ and such that for all $i, j \in \{1, ..., k\}$: $N_i \subset N_j$ if and only if $i < j$. Let $P_1 := N_1$ and for all $i \in \{2, ..., k\}$ let $P_i := N_i \setminus N_{i-1}$. Then $\mathscr{P} := \{P_i : i \in \{1, ..., k\}\}$ is a partition of I. Define $<_1$ on $\mathscr{P}$ by $P_i <_1 P_j$ if and only if $i < j$.

If $x \succ y$ then there is an $i \in \{1, ..., k\}$ such that $|N_i(x, y)| > |N_i(y, x)|$ and $|N_j(x, y)| = |N_j(y, x)|$ for all $j < i$. By definition of P_i it follows that $|N_i(x, y)| = |P_i(x, y)| + |N_{i-1}(x, y)|$ and hence $|P_i(x, y)| = |N_i(x, y)| - |N_{i-1}(x, y)| > |N_i(y, x)| - |N_{i-1}(x, y)| = |N_i(y, x)| - |N_{i-1}(y, x)| = |P_i(y, x)|$. Similarly it follows that $|P_j(x, y)| = |P_j(y, x)|$ for all $j < i$ (and hence $P_j <_1 P_i$). If $x \sim y$ then $|N_i(x, y)| = |N_i(y, x)|$ for all $i \in \{1, ..., k\}$. Hence $|P_i(x, y)| = |N_i(x, y)| - |N_{i-1}(x, y)| = |N_i(y, x)| - |N_{i-1}(y, x)| = |P_i(y, x)|$ for all $i \in \{1, ..., k\}$. □

C.3 Proof of lemma 7.1

Proof. Let $\succ$ be a lexicographic majority rule with partition $\mathscr{P}$. Choose *positive* weights $(w_i)_{i \in I}$ such that $w_i = w_j$ whenever $i, j \in S$ for some $S \in \mathscr{P}$, and such that the inequalities in equation (4) are satisfied. Note that this is possible: Let R be the largest $R \in \mathscr{P}$ with respect to $<_1$. Let $c > 0$ and set $w_i = c$ for all $i \in R$. If $\mathscr{P} = \{R\}$ we are done. Otherwise, let R' be the largest element in $\mathscr{P} \setminus \{R\}$ according to $<_1$. Choose $c' > c|R| > 0$ and set $w_i = c'$ for all $i \in R'$. If $\{R, R'\} = \mathscr{P}$ we are done. Otherwise continue as above. By finiteness of $\mathscr{P}$ the process must eventually stop.

To prove (3) it suffices to show that for all $x, y \in X$:

$$\begin{cases} x \succ y \Longrightarrow \sum_{i \in \mathbf{P}(x,y)} w_i > \sum_{i \in \mathbf{P}(y,x)} w_i, \\ x \sim y \Longrightarrow \sum_{i \in \mathbf{P}(x,y)} w_i = \sum_{i \in \mathbf{P}(y,x)} w_i. \end{cases}$$

Let $x, y \in X$ such that $x \succ y$, then there is an $R \in \mathscr{P}$ such that $|R(x,y)| > |R(y,x)|$ and $|S(x,y)| = |S(y,x)|$ for all $S <_1 R$. Note that $S \cap \mathbf{P}(x,y) = S(x,y)$ for all $S \in \mathscr{P}$. We have:

$$\sum_{i \in \mathbf{P}(x,y)} w_i - \sum_{i \in \mathbf{P}(y,x)} w_i = \sum_{S \in \mathscr{P}} \Big(\sum_{i \in (\mathbf{P}(x,y) \cap S)} w_i - \sum_{i \in (\mathbf{P}(y,x) \cap S)} w_i \Big) \tag{6}$$

$$= \sum_{i \in R(x,y) \setminus R(y,x)} w_i + \sum_{R <_1 S} \Big(\sum_{i \in S(x,y)} w_i - \sum_{i \in S(y,x)} w_i \Big) \tag{7}$$

$$\geq \sum_{R <_1 S} \sum_{i \in S(x,y)} w_i + \Big(w_i - \sum_{R <_1 S} \sum_{i \in S(y,x)} w_i \Big) \tag{8}$$

$$> \sum_{R <_1 S} \sum_{i \in S(x,y)} w_i + 0 \geq 0. \tag{9}$$

The equality in (7) follows since $|S(x,y)| = |S(y,x)|$ for $S \in \mathscr{P}$ such that $S <_1 R$ and since $w_i = w_j$ whenever $i, j \in S$ for some $S \in \mathscr{P}$. The inequality in (8) follows by rearranging terms and since $i \in R$ and $w_i > 0$ for all $i \in I$. The first inequality in (9) follows since the weights were chosen to satisfy (4). The last inequality in (9) follows since $w_i > 0$ for all $i \in I$.

Let $x, y \in X$ such that $x \sim y$. Then since preferences are lexicographic majority: $|S(x,y)| = |S(y,x)|$ for all $S \in \mathscr{P}$. It follows that

$$\sum_{i \in \mathbf{P}(x,y)} w_i = \sum_{S \in \mathscr{P}} \sum_{i \in S(x,y)} w_i = \sum_{S \in \mathscr{P}} \sum_{i \in S(y,x)} w_i = \sum_{i \in \mathbf{P}(y,x)} w_i. \tag{10}$$

By previous paragraphs it hence follows that $x \succ y$ if and only if $\sum_{i \in \mathbf{P}(x,y)} w_i > \sum_{i \in \mathbf{P}(y,x)} w_i$ for all $x, y \in X$. Thus preferences are weighted majority.

For the other direction assume that $\succ$ is a weighted majority rule such that the inequalities in (4) hold for some partition $\mathscr{P}$ and linear order $<_1$. Let $\succ'$ be the lexicographic majority relation with partition $\mathscr{P}$ and linear order $<_1$. It suffices to show that for all $x, y \in X$:

$$\begin{cases} x \succ' y \Longrightarrow x \succ y \\ x \sim' y \Longrightarrow x \sim y \end{cases}$$

Let $x, y \in X$ and $x \succ' y$, then $|R(x,y)| > |R(y,x)|$ for some $R \in \mathscr{P}$ and $|S(y,x)| > |S(x,y)|$ for no $S <_1 R$. Pick the smallest $R \in \mathscr{P}$, according to $<_1$, such that $|R(x,y)| > |R(y,x)|$. Repeating the computations in equation (6)-(9) shows that $\sum_{i \in \mathbf{P}(x,y)} w_i > \sum_{i \in \mathbf{P}(y,x)} w_i$. Hence $x \succ y$.

Finally, let $x, y \in X$ be such that $x \sim' y$ then $|S(x,y)| = |S(y,x)|$ for all $S \in \mathscr{P}$. The calculations in equation (10) imply $\sum_{i \in \mathbf{P}(x,y)} w_i = \sum_{i \in \mathbf{P}(y,x)} w_i$, so $x \sim y$. □

C.4 Proof of necessity of axioms in theorem 4.1

Proof. Let $\succ$ be a lexicographic majority preference with partition $\mathscr{P}$ and linear order $<_1$ on $\mathscr{P}$. We show that $\succ$ satisfies noncompensation A1, restricted negative transitivity A2, majority robustness A3 part a) and majority robustness A3 part b).

Majority robustness: (Part a)) Let $x, y \in X$ be such that $x \succ y$ and $|\mathbf{P}(x,y)| > |\mathbf{P}(y,x)|$. Then we cannot have $|S(x,y)| \leq |S(y,x)|$ for all $S \in \mathscr{P}$, as this would imply that $|\mathbf{P}(x,y)| = \sum_{S\in\mathscr{P}} |S(x,y)| \leq \sum_{S\in\mathscr{P}} |S(y,x)| = |\mathbf{P}(y,x)|$. Hence $|R(x,y)| > |R(y,x)|$ for some $R \in \mathscr{P}$. Pick the smallest such $R \in \mathscr{P}$ with respect to $<_1$ (and note that we cannot have $|S(y,x)| > |S(x,y)|$ for some $S <_1 R$ as that would imply that $y \succ x$). If $|R(x,y)| > 1 + |R(y,x)|$ then there is an $i \in R$ such that $(x_{-i}, z_i) \succsim (y_{-i}, z_i)$ for all $z_i \in X_i$, and we are done. Hence w.l.o.g. $|R(x,y)| = 1 + |R(y,x)|$.

Assume there is an $S \in \mathscr{P}$ such that $R <_1 S$ and $|S(x,y)| > 0$, then there is an $i \in S$, such that $(x_{-i}, z_i) \succsim (y_{-i}, z_i)$ for all $z_i \in X_i$, and we are done again. Thus we may w.l.o.g. assume that $|S(x,y)| = 0$ for all $S \in \mathscr{P}$ with $R <_1 S$. Assume that $|S(y,x)| > 0$ for some $R <_1 S$, then since further $|S(x,y)| = |S(y,x)|$ for all $S <_1 R$, we have

$$|\mathbf{P}(x,y)| = \sum_{S\in\mathscr{P}} |S(x,y)| =$$

$$\sum_{S<_1 R} |S(x,y)| + \sum_{R<_1 S} |S(x,y)| + |R(x,y)| =$$

$$\sum_{S<_1 R} |S(y,x)| + |R(y,x)| + 1 \leq \sum_{S\in\mathscr{P}} |S(y,x)| = |\mathbf{P}(y,x)|,$$

which is a contradiction. Hence we may further reduce to the case where $|S(y,x)| = 0$ for all $S \in \mathscr{P}$ with $R <_1 S$, $|S(x,y)| = |S(y,x)|$ for all $S \in \mathscr{P}$ with $S <_1 R$ and $|R(x,y)| = 1 + |R(y,x)|$. But in such a case, for $i \in R(x,y)$ we have $(z_i, x_{-i}) \sim (z_i, y_{-i})$ for all $z_i \in X_i$ and we are done.

(Part b)) Let $x, y \in X$ have $|\mathbf{P}(x,y)| > |\mathbf{P}(y,x)|$ and $x \sim y$. Then $|R(x,y)| \neq |R(y,x)|$ for some $R \in \mathscr{P}$. Looking at the first such R according to the linear order $<_1$, we see that $x \succ y$ if $|R(x,y)| > |R(y,x)|$ and $y \succ x$ otherwise.

Restricted negative transitivity: Fix a binary product set $X' = \times_{i\in I}\{a_i, b_i\} \subset X$ where $a_i \succ_i b_i$ for all $i \in I$. Let $x, y, w \in X'$ and $x \succ y$. Since $x \succ y$, there is an $R \in \mathscr{P}$ such that $|R(x,y)| > |R(y,x)|$ and $|S(x,y)| = |S(y,x)|$ for all $S \in \mathscr{P}$ with $S <_1 R$. Assume that $w \succsim x$. If $w \sim x$, then $|S(x,w)| = |S(w,x)|$ for all $S \in \mathscr{P}$. Now note that for all $S \in \mathscr{P}$ and $x, y \in X'$ we have $|S(x,y)| > |S(y,x)|$ if and only if $|\{i \in S : x_i = a_i\}| > |\{i \in S : y_i = a_i\}|$. Thus $|S(x,y)| = |S(y,x)|$ together with $|S(x,w)| = |S(w,x)|$, implies that $|\{i \in S : y_i = a_i\}| = |\{i \in S : x_i = a_i\}| = |\{i \in S : w_i = a_i\}|$, and hence $|S(y,w)| = |S(w,y)|$ for all $S <_1 R$. Since $|R(x,y)| > |R(y,x)|$ and

$|R(x,w)| = |R(w,x)|$, implies that $|\{i \in S : w_i = a_i\}| = |\{i \in S : x_i = a_i\}| > |\{i \in S : y_i = a_i\}|$ we have $|R(w,y)| > |R(y,w)|$ and by definition of $\succ$ (as a LM preference) it follows that $w \succ y$. Suppose instead that $w \succ x$, then similar calculations show that $w \succ y$. This shows that $\succ$ is negatively transitive.

Noncompensation: Let $x, y, z, w \in X$ be such that $\mathbf{P}(x,y) = \mathbf{P}(w,z)$ and $\mathbf{P}(y,x) = \mathbf{P}(z,w)$, then $S(x,y) = S(w,z)$ and $S(y,x) = S(z,w)$ for all $S \in \mathscr{P}$, and by definition of lexicographic majority it is clear that $x \succ y$ if and only if $w \succ z$. □

D Appendix: Pareto lemma

Here we state a lemma used in our characterization result 4.1. The lemma is from Petri and Voorneveld (2016), where a proof can also be found. Basically the lemma says that any relation $\succ$ satisfying A2 and A1 extends the Pareto relation, i.e. it ranks every pair of elements $x, y \in X$ that the Pareto relation on X ranks. Note that only transitivity of $\succ$ is used in the proof of (b), hence negative transitivity A2 is stronger than necessary to prove (b).

Lemma D.1 (Pareto). *Let $\succ$ be an asymmetric relation on $X = \times_{i \in I} X_i$.*

(a) Let $\succ$ satisfy noncompensation A1. For all $x, y \in X$, if $x_i \sim_i y_i$ for all $i \in I$: $x \sim y$.

(b) Let $\succ$ satisfy transitivity requirement A2 on $X' = \times_{i \in I}\{a_i, b_i\} \subset X$. For all $x, y \in X'$ with $x_i \succsim_i y_i$ for all $i \in I$ and $x_i \succ_i y_i$ for some $i \in I$: $x \succ y$.

E Appendix: Logical independence

We prove that the axioms used in theorem 4.1 are logically independent. This is proven by four examples, were each example violates exactly one of the axioms.

Proposition E.1. *Axioms A1, A2, A3 part a) and A3 part b) are logically independent.*

The first three examples below are taken from Petri and Voorneveld (2016) and adapted to show independence in the current setting. Example E.4 is introduced to show independence of axiom A3 part b).

Example E.1 (A1 violated). The linear order $\succ$ on $X = \{0,2\} \times \{0,1,4\}$ represented by utility function $u : X \to \mathbb{R}$ with $u(x) = x_1 + x_2$ has

$$(2,4) \succ (0,4) \succ (2,1) \succ (2,0) \succ (0,1) \succ (0,0).$$

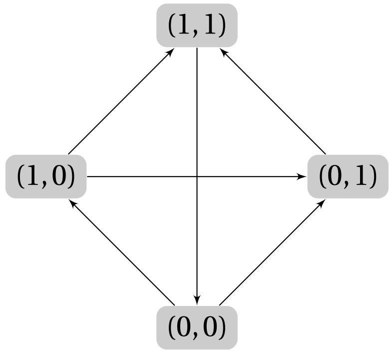

Figure 3: Violation of A2.

So $\succ$ is asymmetric. The utility function is increasing in each coordinate, so $\succ_1$ and $\succ_2$ coincide with the usual order $>$ on integers.

Since $\succ$ is negatively transitive, A2 holds. A3 part a) holds: If $\mathbf{P}(y,x) \neq \emptyset$ then it imposes restrictions only on pair vectors $x, y \in X$ with $\left|\mathbf{P}(x,y) \cup \mathbf{P}(y,x)\right| = 3$, but $|I| = 2$. If $\mathbf{P}(y,x) = \emptyset$ then robustness holds if the conclusions of part a) and part b) of the Pareto lemma hold. But $\succ$ is represented by an additive utility function $u : X \to \mathbb{R}$ and hence $\succ$ extends the Pareto order.

A3 part b) holds: There are no indifferences.

A1 is violated: let $x = w = (2,0), y = (0,1), z = (0,4)$. Then $\mathbf{P}(x,y) = \mathbf{P}(w,z) = \{1\}, \mathbf{P}(y,x) = \mathbf{P}(z,w) = \{2\}$, but $x \succ y$ and $z \succ w$. ◁

Example E.2 (A2 violated)**.** Define a relation $\succ$ on $X = \times_{i=1}^{2}\{0,1\}$ by $x \succ y$ if and only if there is a directed edge from y to x in Figure 3. So $\succ$ is asymmetric. Moreover, $1 \succ_1 0$, since there is an edge from $(0,0)$ to $(1,0)$ and from $(0,1)$ to $(1,1)$. Similarly, $1 \succ_2 0$, since there is an edge from $(0,0)$ to $(0,1)$ and from $(1,0)$ to $(1,1)$.

A1 holds: X has only two factors $X_i = \{0,1\}$ and $1 \succ_i 0$ for both $i = 1,2$.

A3 part a) holds: Since $|I| \leq 2$ it follows that part a) of majority robustness holds if and only if it holds for all vectors $x, y \in X$ with $\mathbf{P}(y,x) = \emptyset$. This is easily seen to be the case.

A3 part b) holds: There are no indifferences.

A2 is violated: $(1,1) \succ (1,0)$ and $(1,0) \succ (0,0)$, but not $(1,1) \succ (0,0)$. ◁

Example E.3 (A3 part a) violated)**.** The linear order $\succ$ on $X = \times_{i=1}^{3}\{0,1\}$ represented by utility function $u : X \to \mathbb{R}$ with $u(x) = 2x_1 + 3x_2 + 4x_3$ has

$$(1,1,1) \succ (0,1,1) \succ (1,0,1) \succ (1,1,0) \succ (0,0,1) \succ (0,1,0) \succ$$
$$(1,0,0) \succ (0,0,0).$$

So $\succ$ is asymmetric. The utility function is increasing in each coordinate, so $1 \succ_i 0$ for all

$i \in \{1,2,3\}$. Let $w = (2,3,4) \in \mathbb{R}^3$. Since $x_i, y_i \in \{0,1\}$ for all $i = 1,2,3$, we can rewrite

$$x \succ y \quad \Leftrightarrow \quad \sum_{i=1}^{3} w_i(x_i - y_i) > 0 \quad \Leftrightarrow \quad \sum_{i \in \mathbf{P}(x,y)} w_i > \sum_{i \in \mathbf{P}(y,x)} w_i.$$

So A1 holds. A2 holds: $\succ$ is negatively transitive.

A3 part b) holds: There are no indifferences.

A3 part a) is violated: $(1,1,0) \succ (0,0,1)$, so A3 implies that $(0,1,0) \succsim (0,0,1)$ (if coordinate $i = 1$ is changed) or $(1,0,0) \succsim (0,0,1)$ (if coordinate $i = 2$ is changed). A contradiction in both cases. ◁

Example E.4 (A3 part b) violated)**.** The linear order $\succ$ on $X = \times_{i=1}^{3}\{0,1\}$ represented by utility function $u : X \to \mathbb{R}$ with $u(x) = 2x_1 + 3x_2 + 5x_3$ has

$$(1,1,1) \succ (0,1,1) \succ (1,0,1) \succ (1,1,0) \sim (0,0,1) \succ (0,1,0) \succ$$
$$(1,0,0) \succ (0,0,0).$$

So $\succ$ is asymmetric. The utility function is increasing in each coordinate, so $1 \succ_i 0$ for all $i \in \{1,2,3\}$. Let $w = (2,3,5) \in \mathbb{R}^3$. Since $x_i, y_i \in \{0,1\}$ for all $i = 1,2,3$, we can rewrite

$$x \succ y \quad \Leftrightarrow \quad \sum_{i=1}^{3} w_i(x_i - y_i) > 0 \quad \Leftrightarrow \quad \sum_{i \in \mathbf{P}(x,y)} w_i > \sum_{i \in \mathbf{P}(y,x)} w_i.$$

So A1 holds. A2 holds: $\succ$ is negatively transitive.

A3 part a) holds. If $x, y \in X$ satisfies the premise of A3 part a), then either $x = (0,1,1)$ and $y = (1,0,0)$ or $x = (1,0,1)$ and $y = (0,1,0)$, and the conclusion of A3 part a) is easily verified to hold in both cases.

part a)

A3 part b) is violated: $(1,1,0) \sim (0,0,1)$ and A3 part b) implies that $(1,0,0) \sim (0,0,1)$ or $(0,1,0) \sim (0,0,1)$, a contradiction.

◁

References

D. Bouyssou and M. Pirlot. A characterization of concordance relations. *European Journal of Operational Research*, 167(2):427–443, 2005.

D. Bouyssou and M. Pirlot. Further results on concordance relations. *European Journal of Operational Research*, 181(1):505–514, 2007.

D. Bouyssou and J.-C. Vansnick. Noncompensatory and generalized noncompensatory preference structures. *Theory and Decision*, 21(3):251–266, 1986.

P. C. Fishburn. Utility theory for decision making. Technical report, DTIC Document, 1970a.

P. C. Fishburn. Intransitive indifference in preference theory: a survey. *Operations Research*, 18(2):207–228, 1970b.

P. C. Fishburn. Axioms for lexicographic preferences. *The Review of Economic Studies*, pages 415–419, 1975.

P. C. Fishburn. Noncompensatory preferences. *Synthese*, 33(1):393–403, 1976.

P. Manzini and M. Mariotti. Choice by lexicographic semiorders. *Theoretical Economics*, 7 (1):1–23, 2012.

K. O. May. A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica*, pages 680–684, 1952.

K. O. May. Intransitivity, utility, and the aggregation of preference patterns. *Econometrica*, pages 1–13, 1954.

H. Petri and M. Voorneveld. Characterizing lexicographic preferences. *Journal of Mathematical Economics*, 63:54–61, 2016.

J. E. Russo and B. A. Dosher. Strategies for multiattribute binary choice. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 9(4):676–696, 1983.

A. Tversky. Intransitivity of preferences. *Psychological Review*, 76(1):31–48, 1969.

M. Yee, E. Dahan, J. R. Hauser, and J. Orlin. Greedoid-based noncompensatory inference. *Marketing Science*, 26(4):532–549, 2007.

J. Zhang, C. K. Hsee, and Z. Xiao. The majority rule in individual decision making. *Organizational Behavior and Human Decision Processes*, 99(1):102–111, 2006.