Characterizing Ordered Models of Behavioral Heterogeneity⋆

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Abstract. The primary objective of the present paper is to demonstrate the applicability of some recent theories of ordered random choice in describing heterogenous choice behaviors. Within the ordered framework, we establish a link between aggregate and individual choice. This link allows us to characterize a wide range of ordered models of behavioral heterogeneity by leveraging on existing characterizations of individual (deterministic) choice. We also show that our results straightforwardly extend to infinite choice spaces enabling applications to choice under risk and uncertainty (expected utility).

1. Introduction

At the core of many economic models is an individual, or a population of individuals, that make choices from a set of alternatives. Typically, the choices made by individuals are heterogenous. Introspection, as well as evidence from diverse fields such as finance (Curcuru, Heaton, Lucas, and Moore, 2010), behavioral economics (Von Gaudecker, Van Soest, and Wengstrom, 2011), discrete choice (Train, 1998) and marketing (Draganska and Klapper, 2011) support this. While constructing a choice model that accommodates heterogeneity is conceptually straightforward, the analysis of such models is often impeded by their large number of parameters.

There are two interconnected issues in studying models of choice heterogeneity: identification and characterization. A choice model is identified if its underlying parameters can be recovered from observed choice data. The second issue, characterization, loosely refers to the relation between *unobserved* individual and *observed* aggregate choice behaviors. For instance, in a population of utility maximizers, how can we charactrize aggregate behavior. I.e. what are the observable implications of the model? Conversely, what properties must aggregate behavior satisfy to be consistent with utility maximization on part of individuals? Both of these issues, identification

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and characterization, are non-trivial to address for models of choice heterogeneity due to their many parameters. While recent attention has been directed towards the identification issue, the characterization issue often remains elusive. This paper aims to present a simple framework to address the characterization issue across a broad range of models of choice heterogeneity.

Towards this aim, we draw on a recent literature on stochastic choices from ordered menus. The model that we consider posits a probability distribution on a set of choice types/individuals. The principal assumption is that underlying (unobservable) types can be ordered in such a way that higher types are more aligned with the menu order than lower types. Besides being technically convenient (i.e. imply identification) the ordering condition on types is also satisfied in a number of economic settings.[1] A contribution of the present paper is to show that not only are many models identified within this setting (as shown in prior work) but they are also straightforward to describe/characterize in terms of (easily interpretable) behavioral properties.

Our first result is a representation result that allows us to establish a link between individual and aggregate behaviors. Readers familiar with the literature on random choice, will notice that our representation result is closely related to recent contributions on ordered random choice models.[2] We wish to stress that the model definition per se should not be viewed as our main contribution. Rather, as we will discuss more later, our contribution is to formulate the model in a way that makes it applicable in a range of relevant settings. In particular, our formulation of the model makes it transparent that the correspondences in the support of the representation explicitly depends on the (observed) empirical primitive. This makes it straightforward to infer properties of (to the researcher unobserved) individual choices using aggregate choices (the observed stochastic choice rule) and allows us to characterize a range of different models of choice heterogeneity. Further, our formulation of the model easily extends to infinite choice domains, that are relevant in many economic environments such as choice under risk/uncertainty and multi-attribute choice.

The main tool in our analysis is the notion of a cumulative stochastic choice function (cumulative SCF).[3] It associates to each alternative a and menu A the probability

[1]See e.g. Apesteguia et al. (2017),Filiz-Ozbay and Masatlioglu (2022) and Petri (2023a) for discussions of economically relevant examples.

[2]See Apesteguia, Ballester, and Lu (2017), Dardanoni, Manzini, Mariotti, Petri, and Tyson (2022), Filiz-Ozbay and Masatlioglu (2022) and Petri (2023a).

[3]In a recent, independent, paper Apesteguia and Ballester (2023a) also consider cumulative stochastic choices as the empirical primitive in a characterization of ordered logit.

of choosing an alternative weakly better than a according to a given order of alternatives. The usefulness of this notion stems from the fact that cumulative SCFs can be used as an alternative empirical primitive in our framework. Knowing the cumulative SCF the researcher can easily describe the underlying SCF, and conversely any SCF gives rise to a cumulative SCF. It turns out that the choice correspondences in the support of our model can be described as the alternatives with cumulative probability exceeding a certain threshold. By imposing suitable conditions on the (observed) cumulative stochastic choice function we can thus characterize more specialized models of behavioral heterogeneity.

We present several applications of the ordered framework. As a first application we establish links between well-known properties of individual (deterministic) choice and corresponding aggregate (stochastic) properties. To illustrate, consider Sen's property α (Sen, 1971). It is a deterministic property and says that if an alternative is chosen in a set/menu and other alternatives are eliminated from the menu, then the alternative should still be chosen in the smaller menu. As an example, if you choose coffee when coffee, tea and chocolate are available you should choose coffee when only coffee and tea are available. We formulate a stochastic version of this property, stochastic property α, and show that a stochastic choice rule satisfies the stochastic version if and only if all the individual correspondences in the support of our model satisfy property α. We perform a similar analysis on other well-known properties of individual choice, such as the classic expansion property (called property γ in Sen (1971)) and various transitivity properties.

We also apply our results to study ordered versions of the random utility model. I.e. models where the choices of individual types are dictated by preference relations. Perhaps noteworthy is that we provide an analysis of incomplete/indecisive preferences within our framework. To the best of our knowledge very few papers explore choice heterogeneity in the context of indecisive/incomplete preferences. We consider a model where individuals use incomplete preferences and are ordered by their degree of indecision. Higher types are less indecisive than lower types. The ordering of alternatives in menus is used as a tie-breaking rule to select a unique alternative whenever there is indecision or indifference between alternatives in menus. The model is characterized by stochastic analogues of the properties characterizing individual choice correspondences maximized by incomplete preferences.

In our second application we consider more structured domains (with a topological structure imposed on the grand set of alternatives). In particular, we characterize expected utility versions of our model. The model is related to the random expected utility model of Gul and Pesendorfer (2006) and allows for heterogeneity in expected utility preferences. The characterization is in terms of straightforward adaptions of the von Neuman Morgenstern axioms (Von Neumann and Morgenstern, 1947) to our setting. We also discuss a special case of this model with risk averse preferences. To the best of our knowledge, characterizing the random expected utility model with risk averse preferences remains an open problem (in particular Gul and Pesendorfer (2006) consider finite prize spaces). We characterize a special case with an ordered type space.

1.1. **Related literature.** Broadly, this paper relates to a literature that studies heterogeneity in choice. Modeling issues related to heterogeneity in choice have received attention in fields such as macroeconomics, econometrics, finance and decision theory. Our paper is particularly close to a subset of the decision theoretic literature that studies variations and extensions of the random utility model (Block and Marschak, 1960; Barberá and Pattanaik, 1986; Falmagne, 1978). It was early acknowledged that the random utility model suffers from identification issues. Scholars have therefore focused on variations of RUM with more structure. Notably, Gul and Pesendorfer (2006) consider an expected utility version of RUM using straightforward adaptions of the von Neumann Morgenstern axioms.[4] Manzini and Mariotti (2018) consider a RUM with limited support, i.e. where the support of the RUM consists of two preferences.[5] Further, Apesteguia, Ballester, and Lu (2017) consider a version of RUM where the collection of preferences in the support of the RUM satisfies the single-crossing condition. In recent contributions, Filiz-Ozbay and Masatlioglu (2022) and Dardanoni et al. (2022) introduce and discuss random choice models. Being extensions of the random utility framework they are subject to the same type of identification issues as the random utility model.

Starting with the work of Fishburn (1978) a literature has emerged that studies means to approximate a given stochastic choice function by deterministic correspondences (see e.g. Ok and Tserenjigmid (2022) and Ok and Tserenjigmid (2021)). Fishburn introduces a class of choice correspondences that are "derived" from the

[4]The papers Lu (2016) and Lu (2021) discuss models of ambiguity aversion in the context of RUM.
[5]See also Manzini, Mariotti, and Petri (2019).

stochastic choice rule under consideration. Formally, for each SCF ρ a parametric family of choice correspondence $(C_\lambda)_{\lambda\in(0,1]}$ is defined, where $C_\lambda(A) = \{a \in A : \rho(a, A) \geq \lambda \max_{b\in A} \rho(b, A)\}$ for all $A \subseteq X$. Similarly to us, Fishburn studies the relationships between properties satisfied by the approximating C_λ correspondences and the SCF ρ. Although our approaches are similar, there are some important conceptual differences. First, every SCF ρ cannot be written in an intuitive manner as a probability distribution on C_λ correspondences. In contrast, we show that every SCF can be described as a (unique) probability distribution on a set of choice correspondences derived from the SCF. There is thus a clear interpretation of the deterministic correspondences in our framework as the choice behaviors of individual types. Interpreting the C_λ correspondences as such is more problematic. Second, we consider infinite choice settings and applications to the important domains of risk and uncertainty, whereas Fishburn (and the literature following Fishburn) mainly considers finite choice environments.

2. Model

We denote by X an arbitrary (finite or infinite) set of alternatives. A nonempty finite subset $A \subseteq X$ is called a *menu*. Let $\mathcal{A}$ denote a collection of nonempty finite subsets of X, i.e. $\mathcal{A}$ is a collection of menus. We wish to emphasize that we put no restrictions on the set X at the outset. This makes our framework applicable to a wide variety of domains, including choice under risk and uncertainty (i.e. choice from prospects/lotteries), convex subsets of $\mathbb{R}^m$, and discrete domains. Applications to these domains will be considered in section 5.

The empirical primitive of our exercise is a stochastic choice function. A stochastic choice function (SCF) is a map $\rho : X \times \mathcal{A} \mapsto [0, 1]$ such that i) $\sum_{a\in A} \rho(a, A) = 1$ for all $A \in \mathcal{A}$ and ii) $\rho(a, A) = 0$ for all $a \in X \setminus A$. An analyst thus observes a probability distribution on choices A for all menus A in the collection $\mathcal{A}$. A choice correspondence is a function $C : \mathcal{A} \to 2^X \setminus \emptyset$ such that $C(A) \subseteq A$ for all $A \in \mathcal{A}$. Let $\mathbb{C}$ denote the set of all choice correspondences.

2.1. Ordered menus.

Ordered menus. We consider a domain of ordered menus, that is, for each $A \in \mathcal{A}$ there is an associated (linear) order $\rhd^A$ of A. Alternatives in each menu $A \in \mathcal{A}$ can thus be ordered as $\{a_1, \dots, a_{|A|}\}$ where $a_i \rhd^A a_j$ if and only if $i > j$. It is important to note that the ordering $\rhd^A$ is allowed to be menu dependent. This makes our framework flexible enough to account for a variety of relevant choice domains. We refer to the literature on ordered choice for several economically relevant examples where this condition is satisfied. In section 7 we will apply our results to choice under risk (and infinite domains). Apesteguia and Ballester (2023a) and Apesteguia and Ballester (2023b) discuss several relevant examples of ordered domains in this setting.

2.2. **Benchmark model.** Our model posits a probability distribution μ on choice correspondences in $\mathbb{C}$. Correspondence C is drawn from $\mathbb{C}$ with probability $\mu(C)$. Conditional on correspondence C being drawn the alternative that maximizes $\rhd^A$ among alternatives in $C(A)$ is chosen. Given a correspondence C let $b[C(A)]$ be the $\rhd^A$ best alternative in $C(A)$, i.e. $b[C(A)] = \max(\rhd^A, C(A))$ for all $A \in \mathcal{A}$.

Definition 2.1. A stochastic choice function ρ has a random choice model (RCM) representation if there is a probability measure μ on $\mathbb{C}$ such that

$$\rho(a, A) = \int_{\mathbb{C}} \mathbf{1}\{a = b[C(A)]\} d\mu(C)$$

for all $a \in A$ and $A \subseteq X$. $\lhd$

Analyzing choice heterogeneity within this model is unfortunately not very practical. In general, random choice models suffer from identifiability issues similar to the well-known identifiability issues of the random utility model. We will therefore focus on a more structured version of the model with an ordered type space.

2.3. **Random ordered model.** We will assume that the choice correspondences in the support of an RCM are totally ordered by set inclusion. It is a permissive condition that allows for a high degree of heterogeneity in tastes/preferences and cognitive characteristics such as attention. The next definition formulates the ordering condition for arbitrary (infinite) collections of choice correspondences. Given two choice correspondences C, C' we write $C \supseteq C'$ ($C \subseteq C'$) if $C(A) \supseteq C'(A)$ ($C(A) \subseteq C'(A)$) for all $A \in \mathcal{A}$.

Definition 2.2. A collection of choice correspondences $\mathcal{C}$ is an *ordered collection* if for all $C, C' \in \mathcal{C}$ we have that $C \subseteq C'$ or $C' \supseteq C$. $\lhd$

The ordering condition limits the degree of heterogeneity tolerated by our model. At the same time, it is a weak condition that is consistent with many plausible behaviors. Several examples where this condition is applicable are discussed in Petri (2023a).

Definition 2.3. A stochastic choice function ρ has a random ordered model (ROM) representation if there is a probability measure μ on an ordered collection[6] $\mathcal{C}$ such that

$$\rho(a, A) = \int_{\mathcal{C}} \mathbf{1}\{a = b[C(A)]\} d\mu(C)$$

for all $a \in A$ and $A \subseteq X$, and such that $\text{supp}(\rho) \supseteq C$ for all $C \in \mathcal{C}$.[7] $\triangleleft$

Several interpretations are compatible with the model. The correspondences C can be interpreted as attention correspondences and the random choice model then models a single (or population of) individual(s) who has heterogenous attention levels as encompassed by the measure μ on $\mathcal{C}$ but with fixed/homogenous preferences. Alternatively, the orders $\rhd^A$ can be viewed as outside reference orders, such as list orders. Individuals then choose the first alternative a appearing in the list with $a \in C(A)$.

3. Cumulative stochastic choice function

This section introduces the notion of a cumulative stochastic choice function. It will serve as a powerful tool in characterizing heterogenous choice behaviors in our subsequent analysis. Formally, given a stochastic choice function ρ, we define a *cumulative stochastic choice function (cumulative SCF)* $\Gamma_\rho : X \times \mathcal{A} \to [0, 1]$ by

$$\Gamma_\rho(a, A) = \begin{cases} \sum_{b: b \trianglerighteq^A a} \rho(b, A) & \text{if } \rho(a, A) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

for all $a \in A$ and $A \in \mathcal{A}$. The restriction of Γ_ρ to alternatives $a \in A$ chosen with positive probability $\rho(a, A) > 0$ is natural. Defined in this way Γ_ρ orders alternatives within menus lexicographically. I.e. it may be the case that $b \rhd^A a$ but $\Gamma_\rho(a, A) > \Gamma_\rho(b, A)$ if b is not chosen in A. In words, the cumulative SCF returns for each chosen alternative a and menu A the probability of choosing an alternative at least as good as a. The following lemma shows that there is a natural interpretation of $\Gamma_\rho(a, A)$ in the ROM framework as the fraction of types who consider alternative a in menu A (i.e. for which $a \in C(A)$).

[6]Formally, we assume that μ is defined on a sigma algebra Σ that is generated by the collection of all events $\{C \in \mathcal{C} : C \supseteq C'\}$. This assumption is needed to generalize some of our results to infinite choice spaces.

[7]We let $\text{supp}(\rho)$ denote the support of ρ defined by $\text{supp}(\rho)(A) = \{a \in A : \rho(a, A) > 0\}$ for all $A \in \mathcal{A}$. The requirement that $C \subseteq \text{supp}(\rho)$ for all $C \in \mathcal{C}$ is necessary for identification. Without this condition the ROM representation of a given SCF ρ would not necessarily be unique (example available upon request). If ρ is a positive stochastic choice function, then this condition is redundant.

Lemma 3.1. *Let $\mathcal{C}$ be the support of a ROM representation of ρ. Then for all $A \in \mathcal{A}$ and $a \in A$ we have:*

$$\Gamma_\rho(a, A) = \int_{\mathcal{C}} \mathbf{1}\{a \in C(A)\} d\mu(C).$$

Lemma 3.1 plays an important role in the subsequent analysis. It provides a relation between the observed data ρ and the unobserved correspondences in $\mathcal{C}$. It will be applied frequently in our main analysis in section 5-7.

Proof. Order the alternatives in $\{a \in A : \rho(a, A) > 0\}$ as $\{a_1, \dots, a_n\}$ where $a_1 \rhd^A \dots \rhd^A a_n$. We have to prove that $\Gamma_\rho(a_i, A) = \int_{\mathcal{C}} \mathbf{1}\{a_i \in C(A)\} d\mu(C)$ and that the sets $\{a_i \in C(A)\}$ are measurable for all i. The proof is by induction on $i \in \{1, \dots, n\}$. As base case let $i = 1$. Since $\rho(a_1, A) > 0$ there is a $C \in \mathcal{C}$ with $a \in C(A)$. We show that $a_1 = \max(\rhd^A, C(A))$ if and only if $a_1 \in C(A)$. By definition of a ROM $C(A) \subseteq \{a_1, \dots, a_n\}$ and the conclusion follows from this. Further, by definition of a ROM $\{a_1 = \max(\rhd^A, C(A))\}$ is measurable and hence so is $\{a_1 \in C(A)\}$. Next, note that $\Gamma_\rho(a_1, A) = \rho(a_1, A) = \int_{\mathcal{C}} \mathbf{1}\{a_1 = \max(\rhd^A, C(A))\} d\mu(C) = \int_{\mathcal{C}} \mathbf{1}\{a_1 \in C(A)\} d\mu(C)$. Hence the base case is true.

As induction hypothesis, assume that the claim holds for all $i \leq m-1$. Let $i = m$. Since $\rho(a_m, A) > 0$ there is a correspondence C with $a_m \in C(A)$ and $a_{m-1} \notin C(A)$. Moreover, for all $C \in \mathcal{C}$ we have $a_m = \max(\rhd^A, C(A))$ if and only if $a_m \in C(A)$ and $a_{m-1} \notin C(A)$. It thus follows that

$$\rho(a_m, A) = \int_{\mathcal{C}} \mathbf{1}\{a_m = \max(\rhd^A, C(A)) d\mu(C) = \int_{\mathcal{C}} \mathbf{1}\{a_m \in C(A), a_{m-1} \notin C(A)\} d\mu(C).$$

Using the previous equality, we hence have

$$\Gamma_\rho(a_m, A) = \Gamma_\rho(a_{m-1}, A) + \rho(a_m, A) =$$

$$= \int_{\mathcal{C}} \mathbf{1}\{a_{m-1} \in C(A)\} d\mu(C) + \int_{\mathcal{C}} \mathbf{1}\{a_m \in C(A), a_{m-1} \notin C(A)\} d\mu(C) =$$

$$= \int_{\mathcal{C}} \mathbf{1}\{a_m \in C(A)\} d\mu(C).$$

Thus, the formula holds for all $a \in A$ and $A \in \mathcal{A}$ with $\rho(a, A) > 0$. If $\rho(a, A) = 0$ then $\Gamma_\rho(a, A) = 0$ and since $\text{supp}(\mu) \supseteq C$ for all $C \in \mathcal{C}$ we have that $a \notin C(A)$ for any $C \in \mathcal{C}$ so $\int_{\mathcal{C}} \mathbf{1}\{a \in C(A)\} d\mu(C) = 0 = \Gamma_\rho(a, A)$. □

4. Representation result

This section presents and briefly discusses the representation result on which our main analysis in section 5-7 is build. The central new feature of our result is that we show that the correspondences in the representation can be chosen to explicitly depend on the empirical primitive through the cumulative stochastic choices. More precisely, we show that each correspondence in a ROM representation can be expressed as a correspondence that for each set of alternatives gives the subset of alternatives with cumulative probabilities exceeding a fixed threshold.

Theorem 4.1. *Let ρ be a stochastic choice function. Then*

(1) ρ has a ROM representation with measure μ on an ordered collection $\mathcal{C}$.
(2) Moreover, $\mathcal{C}$ can be chosen such that there for every $C \in \mathcal{C}$ is an $\alpha(C) \in (0,1]$ with

$$C(A) = \{a \in A : \Gamma_\rho(a, A) \geq \alpha(C)\}.$$

The thrust of theorem 4.1 is that it provides a (handy) method to analyze unobserved individual choices using observed aggregate data. Given information provided by observed data ρ, or equivalently Γ_ρ, the researcher can compute the individual choice correspondences $C \in \mathcal{C}$ in a ROM representation of ρ using theorem 4.1. Conversely, given hypotheses on individual behaviors $C \in \mathcal{C}$, the researcher can use theorem 4.1 to derive testable implications on observed choices.

The proof of theorem 4.1 is delegated to section A.1 in the appendix.[8] We next give a brief proof outline. The idea of proof is to build a parametric family of choice correspondences $(C_\alpha)_{\alpha\in[0,1]}$ and then show that the Lebesgue measure λ on $[0,1]$ induces a representation of the form $\int \mathbf{1}\{a = b[C_\alpha(A)]\}d\lambda$. A change of variables then leads to the desired representation. The correspondences C_α are constructed in an explicit way. As in section 3, we define a cumulative SCF $\Gamma_\rho(a, A)$ as the sum of probabilities of alternatives that are weakly better than a in A and then set $C_\alpha(A) = \{a \in A : \Gamma_\rho(a, A) \geq \alpha\}$. The collection $(C_\alpha)_{\alpha\in[0,1]}$ is clearly ordered and the measure $\mu(C) = \lambda(\{\alpha \in [0,1] : C = C_\alpha\})$ then gives a ROM representation of ρ.

[8] In an independent study Yildiz (2023) considers a similar proof technique.

5. Application 1: Relation between aggregate and individual behaviors on finite domains

In this and the following sections we will apply the representation results in section 4 to study special cases of ROM with more structure on the individual choice correspondences in the support of the model. In particular, we will demonstrate through several examples that the ROM framework provides a natural correspondence between properties satisfied by an aggregate/stochastic choice function ρ and the individual/deterministic correspondences C in its support.

5.1. **Property α.** One of the most basic properties of deterministic choice is Sen's property α. A choice correspondence C satisfies *property* α if for all $a \in B \subseteq A$: if $a \in C(A)$ then $a \in C(B)$. In words, if a is chosen in a larger set then it should be chosen in any smaller set containing it. Property α is a characterizing property of choice correspondences maximized by utility functions. Cleary, if an alternative a maximizes a utility function u in set A then a maximizes u in any smaller set containing a as well. We will next formulate a necessary and sufficient condition on Γ_ρ such that each correspondence in the support of a representing μ satisfies property α. The following property generalizes property α to our setting.

Stochastic property α. A stochastic choice function satisfies *stochastic property* α if $a \in B \subseteq A$ implies that $\Gamma_\rho(a, B) \geq \Gamma_\rho(a, A)$.

Proposition 5.1. *Let ρ be a stochastic choice function. The following statements are equivalent:*

- *(i) ρ has a ROM representation where each correspondence C satisfies property α.*
- *(ii) ρ satisfies stochastic property α.*

Proof. ((ii) $\Rightarrow$ (i)) Assume that Γ_ρ satisfies stochastic property α. Let μ on $\mathcal{C}$ be a ROM representation as in theorem 4.1. Let $C \in \mathcal{C}$. There is an $\alpha \in (0, 1]$ such that $C(A) = \{a \in A : \Gamma_\rho(a, A) \geq \alpha\}$ for all $a \in A$ and $A \in \mathcal{A}$. Let $a \in C(A)$ and $a \in B \subseteq A$, then stochastic property α implies that $\Gamma_\rho(a, B) \geq \Gamma_\rho(a, A) \geq \alpha$ and thus $a \in C(B)$. ((i) $\Rightarrow$ (ii)) Assume that each $C \in \mathcal{C}$ satisfies property α. Let $a \in B \subseteq A$. Since each C satisfies property α we have that $\{C \in \mathcal{C} : a \in C(A)\} \subseteq \{C \in \mathcal{C} : a \in C(B)\}$. Thus $\Gamma_\rho(a, B) = \int_{\mathcal{C}} \mathbf{1}\{a \in C(B)\} d\mu(C) \geq \int_{\mathcal{C}} \mathbf{1}\{a \in C(A)\} d\mu(C) = \Gamma_\rho(a, A)$, where the first and last equality follows by lemma 3.1. □

5.2. **Expansion.** In this subsection we formulate a stochastic version of the classic expansion property (called property γ in (Sen, 1971, p.314)). A choice correspondence C satisfies *expansion* if for all $a \in A \cap B$: if $a \in C(A) \cap C(B)$ then $a \in C(A \cup B)$. Thus, if a is chosen in each of two sets then it must also be chosen in their union. I.e. if you prefer coffee to chocolate when only chocolate is available and coffee to tea (when only tea is available), then you should also prefer coffee when both chocolate *and* tea are available. We next formulate a stochastic version of expansion and show that it characterizes ROMs with support consisting of choice correspondences satisfying expansion.

Stochastic expansion. A stochastic choice function ρ satisfies *stochastic expansion* if and only if $\Gamma_\rho(a, A \cup B) \geq \min\{\Gamma_\rho(a, A), \Gamma_\rho(a, B)\}$ for all $a \in A \cap B$ and $A, B \in \mathcal{A}$.

Proposition 5.2. *Let ρ be a stochastic choice function. The following statements are equivalent:*

(i) ρ has a ROM representation where each correspondence C satisfies expansion.
(ii) ρ satisfies stochastic expansion.

Proof. The implication (ii) $\Rightarrow$ (i) is immediate and follows from part 2 of theorem 4.1. We show that (i) $\Rightarrow$ (ii). Assume that each $C \in \mathcal{C}$ satisfies expansion. We show that Γ_ρ satisfies stochastic expansion. Let $A, B \in \mathcal{A}$ and $a \in A \cap B$. By expansion of each C it follows that $\{C \in \mathcal{C} : a \in C(A \cup B)\} \supseteq \{C \in \mathcal{C} : a \in C(A) \cap C(B)\}$ and hence $\Gamma_\rho(a, A \cup B) = \int_{\mathcal{C}} \mathbf{1}\{a \in C(A \cup B)\} d\mu(C) \geq \int_{\mathcal{C}} \mathbf{1}\{a \in C(A) \cap C(B)\} d\mu(C)$, where the first equality follows by lemma 3.1. It thus suffices to show that $\int_{\mathcal{C}} \mathbf{1}\{a \in C(A) \cap C(B)\} d\mu(C) \geq \min\{\Gamma_\rho(a, A), \Gamma_\rho(a, B)\}$. First note that, since $\mathcal{C}$ is ordered, either $\{C \in \mathcal{C} : a \in C(A)\} \supseteq \{C \in \mathcal{C} : a \in C(B)\}$ or $\{C \in \mathcal{C} : a \in C(A)\} \subseteq \{C \in \mathcal{C} : a \in C(B)\}$. Assume that $\{C \in \mathcal{C} : a \in C(A)\} \subseteq \{C \in \mathcal{C} : a \in C(B)\}$ (the other case follows by similar reasoning). Then it follows that $\{C \in \mathcal{C} : a \in C(A) \cap C(B)\} = \{C \in \mathcal{C} : a \in C(A)\}$. Hence $\int_{\mathcal{C}} \mathbf{1}\{a \in C(A) \cap C(B)\} d\mu(C) = \int_{\mathcal{C}} \mathbf{1}\{a \in C(A)\} d\mu(C) = \min\{\Gamma_\rho(a, A), \Gamma_\rho(a, B)\}$. □

5.3. **Transitivity.** A basic tenet of many models of decision making is the classic transitivity property. Preferences are transitive if for all alternatives such that x is preferred to y and y preferred to z, it is the case that x is preferred to z. Consequently, we will call a choice correspondence C *transitive* if for all $x, y, z \in X$: if $x \in C(xy)$ and $y \in C(yz)$ then $x \in C(xz)$. A myriad of different stochastic transitivity properties are considered in the literature (see e.g. Fishburn (1973)). The most basic version is weak stochastic transitivity which requires that if x is chosen more frequently than y, and y more frequently than z, then x should also be chosen more frequently than z. We here formulate a new stochastic transitivity property expressed in terms of the cumulative SCF Γ_ρ.

Stochastic transitivity. A stochastic choice function ρ satisfies *stochastic transitivity* if and only if $\Gamma_\rho(x,z) \geq \min\{\Gamma_\rho(x,y), \Gamma_\rho(y,z)\}$ for all $x, y, z \in X$.

Proposition 5.3. *Let ρ be a stochastic choice function. The following statements are equivalent:*

(i) ρ has a ROM representation where each correspondence C satisfies transitivity.
(ii) ρ satisfies stochastic transitivity.

Proof. The proof is straightforward and follows similar lines of reasoning as the proof proposition 5.2. A complete proof is available from the author upon request. ☐

Transitivity of a choice correspondence C is equivalent to the existence of a weak order $\succsim$ on X such that $C(xy) = \max(\succsim, xy)$. Considering only observations from binary choice menus the model above is thus closely related to ordered versions of the binary random utility model. Petri (2023b) studies binary SCRUMs, i.e. the single-crossing random utility model restricted to the collection of binary choice menus. The model above is (strictly) more restrictive than binary SCRUM. Indeed, as the following proposition shows, stochastic transitivity (as defined above) implies that ρ is a binary SCRUM (whenever the reference order is menu independent). The proof is straightforward and omitted (available from author upon request).

Corollary 5.4. *Let $\rhd$ be a menu independent reference order. If ρ satisfies stochastic transitivity w.r.t. $\rhd$ then ρ is a binary SCRUM.*

5.4. **Strict stochastic transitivity.** Transitivity is a characterizing property of choice correspondences maximized by complete preference relations. However, it is not necessarily a property satisfied by correspondences maximized by incomplete preferences. If preferences are complete and if $x \in c(xy) = \max(\succsim, xy)$ then this reveals that x is weakly preferred to y. But, if we allow for indecision then $x \in c(xy)$ only reveals that x is not dominated by y. I.e. the decision maker may find x and

y incomparable. To reveal a preference for x over y it is necessary that $x = C(xy)$. A property related to, but strictly weaker than, transitivity of a correspondence C is *strict transitivity* which requires that for all $x, y, z \in X$: if $x = C(xy)$ and $y = C(yz)$ then $x = C(xz)$. This property is a characterizing property of choice correspondences maximized by incomplete preference relations. We will use it when characterizing multi-utility versions of our model. Our stochastic version of this property reads as follows.

Strict stochastic transitivity. A stochastic choice function ρ satisfies *strict stochastic transitivity* if for all $x, y, z \in X$: $\Gamma_\rho(x, y) > \Gamma_\rho(y, x)$ and $\Gamma_\rho(y, z) > \Gamma_\rho(z, y)$ implies that

$$\Gamma_\rho(x, z) \geq \min\{\Gamma_\rho(x, y), \Gamma_\rho(y, z)\}, \tag{1}$$

$$\Gamma_\rho(z, x) \leq \max\{\Gamma_\rho(y, x), \Gamma_\rho(z, y)\}. \tag{2}$$

The intuition behind strict stochastic transitivity is quite simple. If $\Gamma_\rho(x, y) > \Gamma_\rho(y, x)$ then this reveals that $x = C(xy)$ for sufficiently "high" correspondences w.r.t. the $\subseteq$ order. More precisely, $1 - \Gamma_\rho(x, y) = \Gamma_\rho(x, y) - \Gamma_\rho(y, x) = \int_{\mathcal{C}} \mathbf{1}\{x = C(xy)\} d\mu(C)$. Using this revelation relation it is then quite clear that the conclusion of strict stochastic transitivity must hold. The observant reader may also note that equation (2) in the statement of strict stochastic transitivity is not implied by equation (1). The reason behind this is that Γ_ρ is a cumulative probability, so unlike the case for stochastic choice functions ρ we may have $\Gamma_\rho(x, y) + \Gamma_\rho(y, x) \neq \Gamma_\rho(y, z) + \Gamma_\rho(z, y)$.

Proposition 5.5. *Let ρ be a stochastic choice function. The following statements are equivalent:*

(i) ρ has a ROM representation where each correspondence C satisfies strict transitivity.

(ii) ρ satisfies strict stochastic transitivity.

Proof. ((ii) $\Rightarrow$ (i)) Assume that ρ satisfies strict stochastic transitivity. Let μ on $\mathcal{C}$ be a ROM representation as in theorem 4.1. Let $C \in \mathcal{C}$. There is then an $\alpha \in (0, 1]$ such that $C(A) = \{a \in A : \Gamma_\rho(a, A) \geq \alpha\}$ for all $a \in A$ and $A \in \mathcal{A}$. We show that each C satisfies strict transitivity. Let $x, y, z \in X$ be such that $x = C(xy)$

and $y = C(yz)$. Since $x = C(xy)$ we have $\Gamma_\rho(x,y) \geq \alpha > \Gamma_\rho(y,x)$ and since $y = C(yz)$ we have $\Gamma_\rho(y,z) \geq \alpha > \Gamma_\rho(z,y)$. Strict stochastic transitivity then implies that $\Gamma_\rho(x,z) \geq \min\{\Gamma_\rho(x,y), \Gamma_\rho(y,z)\} \geq \alpha > \max\{\Gamma_\rho(y,x), \Gamma_\rho(z,y)\} \geq \Gamma_\rho(z,x)$ and hence $C(xz) = x$. ((i) $\Rightarrow$ (ii)) Conversely, assume that each correspondence C satisfies strict transitivity. Let $x, y, z \in X$ with $\Gamma_\rho(x,y) > \Gamma_\rho(y,x)$ and $\Gamma_\rho(y,z) > \Gamma_\rho(z,y)$. Then $\Gamma_\rho(x,y) = 1$ and $\Gamma_\rho(y,z) = 1$. Lemma 3.1 implies that $x \in C(xy)$ and $y \in C(yz)$ for almost all $C \in \mathcal{C}$. Strict transitivity of each C implies that $z \in C(xz)$ for almost all $C \in \mathcal{C}$. Thus $\Gamma_\rho(x,z) = 1 \geq \min\{\Gamma_\rho(x,y), \Gamma_\rho(y,z)\}$. Next, note that for all $x, y \in X$: $\int_\mathcal{C} \mathbf{1}\{x = C(xy)\} d\mu(C) = \Gamma_\rho(x,y) - \Gamma_\rho(y,x) = 1 - \Gamma_\rho(y,x)$. Since $\mathcal{C}$ is ordered we have that either $\{C \in \mathcal{C} : x = C(xy)\} \subseteq \{C \in \mathcal{C} : y = C(yz)\}$ or $\{C \in \mathcal{C} : x = C(xy)\} \supseteq \{C \in \mathcal{C} : y = C(yz)\}$. Assume that $\{C \in \mathcal{C} : x = C(xy)\} \subseteq \{C \in \mathcal{C} : y = C(yz)\}$ (the other case follows by similar reasoning). Then strict transitivity of each $C \in \mathcal{C}$ implies that $\{C \in \mathcal{C} : x = C(xy)\} \subseteq \{C \in \mathcal{C} : x = C(xz)\}$. Hence, it follows that $\min\{1 - \Gamma_\rho(y,x), 1 - \Gamma_\rho(z,y)\} = \int_\mathcal{C} \mathbf{1}\{x = C(xy)\} d\mu(C) \leq \int_\mathcal{C} \mathbf{1}\{x = C(xz)\} d\mu(C) = 1 - \Gamma_\rho(z,x)$. □

6. Application 2: Ordered random multi-utility models

Rationalizable models are models that can be described as resulting from the act of preference maximization. Various conditions can be imposed on the maximizing preference relation such as being acyclic, transitive (but incomplete) and complete. Formally, a choice correspondence is *rationalizable* if there is an acyclic binary relation $\succsim$ on X such that $C(A) = \{x \in A : y \succ x \text{ for no } y \in A\}$ for all $A \in \mathcal{A}$. Similarly, a ROM will be called rationalizable if each correspondence in its support is rationalizable. Sen (1971) shows that a choice correspondence is rationalizable if and only if it satisfies property α and expansion. The following result is hence a direct corollary to proposition 5.1 and proposition 5.2.

Proposition 6.1. *Let ρ be a stochastic choice function. The following statements are equivalent:*

(i) ρ has a ROM representation that is rationalizable (by acyclic relations).
(ii) ρ satisfies stochastic property α and stochastic expansion.

Acyclicity is a weak requirement. A common foundational assumption in economics is that of transitive preferences. We next consider rationalizable ROMs where this stronger requirement on preferences is imposed. A large literature studies incomplete preferences, arguing that completeness is not necessarily a rationality trait (but transitivity is maintained). An individual may fail to compare several pairs of alternatives, but still make consistent choices (in the sense of transitivity). A comprehensive discussion of incomplete preferences is in Eliaz and Ok (2006) and references therein. An incomplete preference relation is a reflexive and transitive binary relation $\succsim$ on X. For a collection of utility functions $\mathcal{U}$ define the multi-utility of alternative $x \in X$ as $\mathcal{U}(x) = (u(x))_{u \in \mathcal{U}}$. A choice rule is maximized by a multi-utility function if there is a collection $\mathcal{U}$ of utility functions such that $C(A) = \{x \in A : \mathcal{U}(y) > \mathcal{U}(x) \text{ for no } y \in A\}$ for all menus $A \subseteq X$.[9] Every incomplete preference relation $\succsim$ has a multi-utility representation.

A stochastic choice function ρ has an *ordered Random Multi-Utility representation* if it has a ROM representation such that each choice rule C is maximized by a multi-utility function (or equivalently an incomplete preference relation). Our next characterization result follows by adapting the characterization result in Ribeiro and Riella (2017) to our setting.

Proposition 6.2. *(Ribeiro and Riella (2017)) A choice correspondence C satisfies property α, expansion and strict transitivity if and only if C is rationalizable by an incomplete preference relation $\succsim$.*

Proposition 6.3. *Let ρ be a stochastic choice function. The following statements are equivalent:*

(i) ρ has an ordered Random Multi-Utility representation.

(ii) ρ satisfies stochastic property α, stochastic expansion and strict stochastic transitivity.

The strongest notion of rationalizability considered in this paper is through a complete preference relation. In finite choice environments this notion is equivalent to rationalizability by a utility function, i.e. the existence of a utility function $u : X \to \mathbb{R}$ such that $C(A) = \arg\max_{a \in A} u(a)$ for all $A \subseteq X$. A ROM ρ with distribution μ has an *ordered random utility model (oRUM)* representation if each correspondence in its support is rationalized by a utility function.

[9] Where $\mathcal{U}(y) > \mathcal{U}(x)$ holds if $u(y) > u(x)$ for some $u \in U$ and $u(y) \geq u(x)$ for all $u \in U$.

Proposition 6.4. *Let ρ be a stochastic choice function. The following statements are equivalent:*

(i) ρ has an oRUM representation.
(ii) ρ satisfies stochastic property α, stochastic expansion and stochastic transitivity.

7. Application 3: Relation between aggregate and individual behaviors on structured domains

In this section we will consider applications to infinite choice spaces endowed with a topological structure. In particular, we will maintain the assumption that X is a *separable metric space* throughout this section. This will facilitate the study of continuous correspondences and applications to the important domains of choice under risk and uncertainty.

7.1. **Continuous choice.** We here consider continuity properties of the correspondences in a ROM representation. A choice correspondence is *continuous* if for any convergent sequences (x_m) and (y_m) in X if $x_m \in C(x_m y_m)$ for all $m \geq 1$ then $x^* \in C(x^* y^*)$ (where $x^* = \lim_{m\to\infty} x_m$ for convergent sequences (x_m)). The following is a natural generalization of continuity to our setting.

Stochastic continuity. A stochastic choice function ρ satisfies *stochastic continuity* if for any convergent sequences (x_m) and (y_m) in X:

$$\limsup_{m\to\infty} \Gamma_\rho(x_m, y_m) \leq \Gamma_\rho(x^*, y^*).$$

The following standard lemma will be needed in the proof of our characterization result below. Let $(E_n)_{n=1}^{\infty}$ be a sequence of subsets of $\mathcal{C}$ and define $\limsup_{m\to\infty} E_m = \bigcap_{m\geq 1} \bigcup_{k\geq m} E_k$.

Lemma 7.1. *For any sequence $(E_n)_{n=1}^{\infty}$ of subsets of $\mathcal{C}$ we have that*

$$\mu(\limsup_{m\to\infty} E_m) \geq \limsup_{m\to\infty} \mu(E_m).$$

Proof. The proof follows by downward continuity of countably additive (probability/finite) measures (Aliprantis and Border, 2006, Theorem 10.8, p.376). □

Proposition 7.2. *Let ρ be a stochastic choice function. The following statements are equivalent:*

(i) ρ has a ROM representation where each correspondence C is continuous.
(ii) ρ satisfies stochastic continuity.

Proof. ((ii) $\Rightarrow$ (i)) Assume that ρ is continuous. Let μ on $\mathcal{C}$ be a ROM representation as in theorem 4.1. Let $C \in \mathcal{C}$. There is then an $\alpha \in (0,1]$ such that $C(A) = \{a \in A : \Gamma_\rho(a, A) \geq \alpha\}$ for all $a \in A$ and $A \in \mathcal{A}$. Let $(x_m), (y_m)$ be convergent sequences such that $x_m \in C(x_m y_m)$ for all $m \geq 1$. Then $\Gamma_\rho(x_m, y_m) \geq \alpha$ for all $m \geq 1$. Since ρ is continuous it follows that $\Gamma_\rho(x^*, y^*) \geq \limsup_{m\to\infty} \Gamma_\rho(x_m, y_m) \geq \alpha$. Hence $x^* \in C(x^* y^*)$. ((i) $\Rightarrow$ (ii)) Conversely, assume that each C is continuous. Let (x_k) and (y_k) be convergent sequences in X. Define for each $k \geq 1$ the set $E_k = \{C \in \mathcal{C} : x_k \in C(x_k, y_k)\}$. Let $E = \bigcap_{m\geq 1} \bigcup_{k\geq m} E_k$. We next show that $E \subseteq \{C \in \mathcal{C} : x^* \in C(x^* y^*)\}$. Let $C \in E$. We will construct (convergent) subsequences (x_{m_k}) and (y_{m_k}) such that $x_{m_k} \in C(x_{m_k} y_{m_k})$ for all $k \geq 1$. Since $C \in \bigcup_{k\geq 1} E_k$, there is an m_1 such that $x_{m_1} \in C(x_{m_1} y_{m_1})$. Assume that $x_{m_1}, \ldots, x_{m_l}$ have been constructed such that $m_1 < \cdots < m_l$ and $x_{m_i} \in C(x_{m_i} y_{m_i})$ for all $1 \leq i \leq l$. Since $C \in \bigcup_{k\geq m_l+1} E_k$ there is a $k \geq m_l + 1$ such that $C \in E_k$ and hence $x_k \in C(x_k y_k)$. Take $m_{l+1} = k$ and the construction is complete. Since $x_{m_k} \in C(x_{m_k} y_{m_k})$ for all $k \geq 1$ and (x_{m_k}) and (y_{m_k}) converge to x^* and y^* respectively, continuity of C implies that $x^* \in C(x^* y^*)$. Lemma 3.1 now implies that $\Gamma_\rho(x^*, y^*) = \int_{\mathcal{C}} \mathbf{1}\{x^* \in C(x^* y^*)\} d\mu(C) \geq \mu(E) \geq \limsup_{k\to\infty} \mu(E_k) = \limsup_{k\to\infty} \Gamma_\rho(x_k, y_k)$, where the inequality $\mu(E) \geq \limsup_{k\to\infty} \mu(E_k)$ follows by lemma 7.1. $\square$

Definition 7.3. A choice correspondence C on X is maximized by a continuous utility function if there is a continuous utility function $u : X \to \mathbb{R}$ such that

$$C(A) = \arg\max_{a\in A} u(a)$$

for all $A \subseteq X$. $\triangleleft$

Lemma 7.4. *A choice correspondence C on X satisfies property α, expansion, transitivity and continuity if and only if it is maximized by a continuous utility function.*

Proof. By previous results we know that there is a complete and transitive relation $\succsim$ that rationalizes C. Since X is a separable metric space (hence second countable), Debreu's representation theorem (Debreu, 1954, Theorem II, p. 163) implies that $\succsim$ has a continuous utility representation. $\square$

We say that a ROM ρ with distribution μ has a *continuous ordered random utility model (continuous oRUM)* representation if each correspondence in its support is maximized by a continuous utility function. The following result is now a direct corollary to proposition 7.2 and proposition 6.1.

Corollary 7.5. *A stochastic choice function ρ has a continuous oRUM representation if and only if ρ satisfies stochastic property α, stochastic expansion, stochastic transitivity and stochastic continuity.*

7.2. **Random expected utility.** Let P be a separable metric space of (certain) prizes and let X be the set of Borel probability measures $\mathcal{M}(P)$ on P endowed with the topology of weak convergence. Let $\mathcal{A}$ be the collection of all finite subsets of X. For all $\lambda \in (0,1)$, the λ mixture of two menus A, B is the menu $\lambda A + (1-\lambda)B = \{\lambda x + (1-\lambda)y : x \in A, y \in B\}$, which is clearly in $\mathcal{A}$ whenever $A, B \in \mathcal{A}$. If $B = \{y\}$ is singleton, the λ mixture of A and $\{y\}$ is with slight abuse of notation denoted $\lambda A + (1-\lambda)y$. The next property on (stochastic) choice rules is a choice theoretic version of the independence axiom (Von Neumann and Morgenstern, 1947). A choice correspondence C satisfies *independence* if for all menus $A \subseteq X$ for all $\lambda \in (0,1)$: $x \in C(A)$ if and only if $\lambda x + (1-\lambda)y \in C(\lambda A + (1-\lambda)y)$. Independence thus requires that if x is chosen in A then a λ mixture of x and y should also be chosen whenever we replace menu A with a menu of λ mixtures of alternatives $x \in A$ and y. The following is a stochastic version of this property, requiring the choice frequency to remain unchanged by λ mixtures with some fixed lottery y.

Stochastic independence. A stochastic choice function ρ satisfies stochastic independence if

$$\Gamma_\rho(\lambda x + (1-\lambda)y, \lambda A + (1-\lambda)y) = \Gamma_\rho(x, A)$$

for all $x \in A \subseteq X$ and $y \in X$ and for all $\lambda \in (0,1)$.

Proposition 7.6. *Let ρ be a stochastic choice function. The following statements are equivalent:*

(i) ρ has a ROM representation where each correspondence C satisfies independence.

(ii) ρ satisfies stochastic independence.

The proof of proposition 7.6 is immediate and hence omitted. Given a Bernoulli (continuous) utility function $u : P \to \mathbb{R}$, define the expected utility of prospect $x \in X$ as $Eu(x) = \int_P u\, dx$. A choice rule C is maximized by an expected utility function if there is a continuous utility function $u : P \to \mathbb{R}$ such that $C(A) = \{x \in A : Eu(x) \geq Eu(y) \text{ for all } y \in A\}$ for all menus $A \subseteq X$. A stochastic choice function ρ has an ordered Random Expected Utility (oREU) representation if it has a ROM representation with each C in its support maximized by an expected utility function.

Proposition 7.7. *A stochastic choice function ρ has an oREU representation if and only if it satisfies stochastic property α, stochastic expansion, stochastic transitivity, stochastic continuity and stochastic independence.*

Proof. Necessity of the properties is straightforward to check. We next sketch a proof for sufficiency. By previous analysis it follows that each C_α satisfies property α, expansion, transitivity, independence and continuity. Define a relation $\succsim_\alpha$ on X by $x \succsim_\alpha y$ if and only if $x \in C_\alpha(xy)$. The resulting relation is then complete, transitive, continuous and satisfies independence. By the von Neumann-Morgenstern utility theorem in Hara, Ok, and Riella (2019, p.939-940) it follows that $\succsim_\alpha$ has an expected utility representation.[10] Using property α, expansion and transitivity of C_α we can then show that $\succsim_\alpha$ maximizes C_α. This completes the proof. □

Risk aversion. We next characterize a version of the oREU model with risk averse preferences. For a given prospect $x \in X$ let $e(x) := \int_P p\, dx$ denote its expected value. Further, for each outcome/prize $p \in P$ let δ_p denote the Dirac measure on P, i.e. $\delta_p(A) = 1$ if $p \in A$ and $\delta_p(A) = 0$ otherwise. A choice correspondence C is *concave* if for all $x \in X$ we have that $\delta_{e(x)} \in C(x\delta_{e(x)})$. The motivation behind this property is that a choice correspondence maximized by expected utility preferences is concave if and only if it is consistent with risk aversion (i.e. has a concave Bernoulli utility function).

Stochastic concavity. A stochastic choice function ρ satisfies stochastic concavity if

$$\Gamma_\rho(\delta_{e(x)}, x) \geq \Gamma_\rho(x, \delta_{e(x)})$$

for all $x \in X$.

[10]The general version of the von Neumann-Morgenstern theorem (with an infinite prize space) stated in Hara, Ok, and Riella (2019) first appears in Grandmont (1972).

Proposition 7.8. *Let ρ be a stochastic choice function. The following statements are equivalent:*

(i) ρ has a ROM representation where each correspondence C satisfies concavity.
(ii) ρ satisfies stochastic concavity.

We omit the straightforward proof of proposition 7.8. An immediate corollary to proposition 7.8 is the following characterization of oREU with risk averse preferences.

Corollary 7.9. *A stochastic choice function ρ has an oREU representation where each type has risk averse preferences if and only if it satisfies stochastic property α, stochastic expansion, stochastic transitivity, stochastic continuity, stochastic independence and stochastic concavity.*

Proof. By proposition 7.7 ρ has a ROM representation where each correspondence C is maximized by an expected utility function u_C. Further, for all $x, y \in X$:

$$x \in C(xy) \quad \Leftrightarrow \quad Eu_C(x) \geq Eu_C(y). \tag{3}$$

By stochastic concavity and proposition 7.8 each correspondence in the support of the ROM satisfies concavity. Hence for all $C \in \mathcal{C}$ and $x \in X$ we have $\delta_{e(x)} \in C(x\delta_{e(x)})$. By equation (3) this implies that for all $x \in X$ we have $u_C(e(x)) = Eu_C(\delta_{e(x)}) \geq Eu_C(x)$. I.e. each u_C is concave. The converse is immediate. □

Appendix A. Proofs

A.1. **Proof of theorem 4.1.** The following standard lemma will be used in the proof of 4.1. We will use it to "push forward" the Lebesgue algebra on $[0, 1]$ to a sigma algebra on $\mathcal{C}$.

Lemma A.1. *Let Σ be a sigma algebra on X and $f : X \to Y$ a function. Then $f \circ \Sigma = \{A \subseteq Y : f^{-1}(A) \in \Sigma\}$ is a sigma algebra on Y.*

The proof of lemma A.1 is standard and hence omitted. We next give the proof of theorem 4.1.

Proof. Let λ be the Lebesgue measure on $[0, 1]$ and $\mathcal{L}$ the Lebesgue algebra. Let $C_0(A) = \{a \in A : \rho(a, A) > 0\}$ and $C_\alpha(A) = \{a \in A : \Gamma_\rho(a, A) \geq \alpha\}$ for all $\alpha \in (0, 1]$ and $A \in \mathcal{A}$. Set $\mathcal{C} = (C_\alpha)_{\alpha \in [0,1]}$ and define a mapping $f : [0, 1] \to \mathcal{C}$ by $f(\alpha) = C_\alpha$ for all $\alpha \in [0, 1]$. Let $f \circ \mathcal{L}$ be the pushforward σ algebra $\{\mathcal{D} \subseteq \mathcal{C} : f^{-1}(\mathcal{D}) \in \mathcal{L}\}$. By construction $\mathcal{C}$ is ordered. Define a measure μ on $f \circ \mathcal{L}$ by $\mu(\mathcal{D}) = \lambda(f^{-1}(\mathcal{D}))$ for all $\mathcal{D} \in f \circ \mathcal{L}$. We claim that ρ has a ROM representation with measure μ. Let $a \in A$ and $\rho(a, A) > 0$ we shall first prove that

$$\rho(a, A) = \lambda(\{\alpha \in [0,1] : a = b[C_\alpha(A)]\}) = \int \mathbf{1}\{a = b[C_\alpha(A)]\} d\lambda.$$

Lemma A.2. *For all $\alpha \in (0,1]$ we have that $a = b[C_\alpha(A)]$ if and only if $\Gamma_\rho(a, A) \geq \alpha > \Gamma_\rho(b, A)$ for all $b \in A$ with $\Gamma_\rho(b, A) > 0$ and $b \rhd^A a$.*

Proof. The proof is straightforward but we provide one for completeness. If $a = b[C_\alpha(A)] = \max(\rhd^A, C_\alpha(A))$ then $\Gamma_\rho(a, A) \geq \alpha$. Assume, by contradiction that there is a $b \in A$ with $b \rhd^A a$ and $\Gamma_\rho(b, A) \geq \alpha$. Then $b \in C_\alpha(A)$ and $b \rhd^A a$ which is a contradiction to $a = \max(\rhd^A, C_\alpha(A))$. Conversely, if $\Gamma_\rho(a, A) \geq \alpha > \Gamma_\rho(b, A)$ for all $b \in A$ with $\Gamma_\rho(b, A) > 0$ and $b \rhd^A a$. Then $a \in C_\alpha(A)$ but there is no $b \in A$ with $b \in C_\alpha(A)$ and $b \rhd^A a$, i.e. $a = \max(\rhd^A, C_\alpha(A)) = b[C_\alpha(A)]$. □

Let b^* be the minimum alternative in A w.r.t. $\rhd^A$ such that $\Gamma_\rho(b, A) > 0$ and $b \rhd^A a$. By lemma A.2 it follows that $\lambda(\{\alpha \in [0,1] : a = b[C_\alpha(A)]\}) = \lambda(\{\alpha \in (0,1] : a = b[C_\alpha(A)]\}) = \lambda(\{\alpha \in (0,1] : \Gamma_\rho(a, A) \geq \alpha > \Gamma_\rho(b^*, A)\}) = \Gamma_\rho(a, A) - \Gamma_\rho(b^*, A) = \rho(a, A)$. By a change of variables (see theorem 13.46 of Aliprantis and Border (2006)) it then follows that $\rho(a, A) = \lambda(\{\alpha \in [0,1] : a = b[C_\alpha(A)]\}) = \int_{\mathcal{C}} \mathbf{1}\{a = b[C(A)]\} d\mu(C)$ for all $a \in A$ and $A \in \mathcal{A}$. □

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